A Market Based Solution for Fire Sales and Other Pecuniary Externalities*

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Abstract

We show how bundling, exclusivity and additional markets internalize fire sales and other pecuniary externalities. Ex ante competition can achieve a constrained efficient allocation. The solution can be put rather simply: create segregated market exchanges which specify prices in advance and price the right to trade in these markets so that participant types pay, or are compensated, consistent with the market exchange they choose and that type's excess demand contribution to the price in that exchange. We do not need to identify and quantify some policy intervention. With the appropriate ex ante design we can let markets solve the problem.

Keywords: price externalities; segregated exchanges; Walrasian equilibrium; markets for rights to trade; market-based solution; collateral; exogenous incomplete markets; fire sales.

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1 Introduction

Our paper is a tale of several literatures and the importance of bringing them together. One in the wake of the financial crisis is a literature on pecuniary externalities that has regained the interest of researchers as they seek policy interventions and regulations to remedy externality-induced distortions, e.g., balance sheet effects, amplifiers and fire sales. Solutions range from regulation of portfolios, restrictions on saving or credit, interest rate restrictions, fiscal policy, or taxes and subsidies levied by the government.[1] A second literature in the general equilibrium tradition is the exogenous incomplete security markets literature, which shows generically that competitive equilibria are inefficient. However, there is a third literature in general equilibrium theory which dates back to work of one of the founding fathers, Arrow (1969), namely how bundling, exclusivity and suitably designed additional markets can internalize externalities, without the need of further policy interventions (or the need to quantify those interventions). In this world, ex ante competition and equilibrium with market-determined prices for rights to trade in these additional markets can achieve a constrained-efficient allocation. The contribution of our paper can be seen as bridging the gaps among these literatures and, more importantly, formulating a proposal for an ex ante optimal market design of financial markets which eliminates fire sales inefficiencies and other pecuniary externalities.

As pointed out by Lorenzoni (2008) and many others[2], both developed and emerging economies have experienced episodes of rapid credit expansion followed, in some cases, by a financial crisis, with a collapse in asset prices, credit, and investment. There is also a literature on fire sales in New York financial markets, e.g, Begalle et al. (2013); Duarte and Eisenbach (2014); Gorton and Metrick (2012); Krishnamurthy et al. (2012). However, as Lorenzoni (2008) emphasizes, if the private sector had accurate expectations and correctly incorporated risk in its optimal decisions, yet still decided to borrow heavily during booms, it means that the expected gain from increased investment more than compensated for the

[1]See, for example, Bianchi (2010); Bianchi and Mendoza (2012); Farhi et al. (2009); Jeanne and Korinek (2013); Korinek (2010).

[2]For the main stylized facts on boom-bust cycles, see Bordo and Jeanne (2002); Gourinchas et al. (2001); Tornell and Westermann (2002), and many others.

expected costs of financial distress. Thus one needs to understand how, and under what conditions, this private calculation leads to inefficient decisions at the social level.

One of our two illustrative examples is a classic environment with incomplete securities, where spot markets are essential. Further, securities need not be state dependent, so a given trade in securities ex ante can have implications for the distribution of income across states; with insufficient ways to hedge, this is precisely why a standard incomplete market equilibrium can be generically inefficient (e.g., Geanakoplos and Polemarchakis, 1986; Greenwald and Stiglitz, 1986). Here, to remove the externality[3], we expand the types of markets. We proceed in three steps.

First we simply allow agents to choose ex ante the price at which they would like to trade ex post; ex ante they are assessing their current security holdings and these future trading actions. This choice of the price is a price vector with components as future spot prices, hence with dimension equal to the number of states. Agents can choose only one price vector, and there is commitment to the choice and excludability. We allow in principle many possible vectors from which an agent can choose, hence many possible segregated exchanges. In effect agents are buying the vector of prices.

Second, and related, there are market exchange fees. That is, agent types are buying the rights to trade at their chosen vector of spot prices. This is how agents pay for the price chosen. That is how the market exchange fees are determined. Alternatively, they may be compensated if the price chosen is adverse to their situation. We specify the quantities of rights to trade as simply, and naturally, a type's vector of excess demands for goods in spot markets for each state s given the chosen segregated exchange. There is a common price per unit excess demand, though the price depends on the segregated exchange chosen. Obviously the quantities of excess demand vary over agent types in any segregated exchange.

Finally security trades are tied to this choice of the price vector; promises to pay or receive security returns must be entered into in an ex ante security market associated with the segregated exchange and executed in the chosen spot markets of that segregated exchange. These rights to trade, securities, and commodities in spot markets are all priced so that in equilibrium the markets in exchanges, rights, and spot markets clear. There are active

[3]We do not expand the set of securities. We do not complete the markets of securities.

exchanges with trade in equilibrium that clear in the usual sense, with the demand and supply of rights equal to each other. There are many inactive exchanges in equilibrium with no trade, chosen by no one. But their potential availability is part of internalizing the externality, is what steers the market for exchanges and securities to a constrained optimum.

For our second leading example, we let prices enter into collateral constraints. Promises in securities must be backed by collateral, carried over to the next period, and the willingness/ability to pay in any given state depends on the spot price of that collateral in that state. Generically competitive equilibria are constrained inefficient, as price consequences are not taken into account. In general aggregate excess demand can be a high dimensional and complex mapping from the entire array of individual pre trade endowments, but with identical homothetic utilities, these ex post spot prices are determined by pre-trade ratios of commodity aggregates, including collateral in escrow being brought into the spot market. In this example then we can naturally refer to this commodity ratio as a market fundamental, as that ratio alone determines prices, and with this get analytical solutions that provide intuition, with agent types in active markets being on one side or another of the fundamental. The collateral economy with a complete set of Arrow-Debreu securities is also special in that rights and the price of rights can be determined state by state, as scalars.

We can relate our solution to the key insights of Coase (1960), Lindahl (1958), Bewley (1981), and Arrow (1969). Coase featured an example with pollution; the allocation of rights does not matter as long as the externality is priced. For us, the externality is two sided, with the position of an agent having to do with that agent type's influence on the price. Some agent types are on one side with excess demand and some on the other with excess supply; net positions must sum to zero.

With Lindahl's pricing of public goods, each agent can buy the amount of the public good they want at an agent specific price. Under individual agent maximization, that agent's price is equal to that agent's marginal utility gain. A producer of the public good then maximizes revenue as the sum of the per unit prices times the quantity produced less production cost. This yields the sum of marginal utilities equal to the marginal cost, an optimum, achieved in a decentralized market. The amount of the public good is common to all of the agents, but there are agent-specific prices. In contrast, in what we do, the relevant quantity is the agent-specific excess demand, which can be positive, negative, or zero, which can vary over agent types, and in equilibrium must sum to zero.[4] The price per unit excess demand is common, not agent specific, but quantities demanded or supplied are agent specific. Bewley (1981) rigorously establishes Tiebout's result but argues that it requires special assumptions and that is not true in general. Lindahl prices can work but in his view they eliminate the essential part of the public goods problem. We note in particular that without Lindahl pricing, without taxes, and with the quantity of public goods in common, expenditure must be the same across agent types. For us, prices are the same across types, so no Lindahl pricing, but quantities, the excess demands, vary because excess demands are different across different types.

We can now go back to Arrow (1969) for his basic fundamental insights about how to remedy externality problems. He deals directly with the most obvious specification of a non-pecuniary externality, preferences that depend on what others are consuming. Arrow extends the commodity space letting each given agent have the right to specify the consumption of the others, as if buying the consumption of others, so that as far as the given agent is concerned, this looks like a normal private ownership economy without externalities. There must also be in equilibrium a consistency in assignments: what every single agent wants some other target agent to consume is exactly the same, what all of them want for that agent, and is what that target agent is actually consuming. Both this and the usual resource constraints pick up Lagrange multipliers in the centralized planning problem. The first in the decentralized market solution is a price for the right of a given agent to specify consumption of a given commodity for another named agent, and the second is the common price of the underlying commodity. We are similar to Arrow in allowing agents to buy rights to trade at prices they choose, in effect specifying something which impacts the others, and to buy

[4]This object is related to consumption rights in Bisin and Gottardi (2006), which internalizes the consumption externality due to adverse selection problem. The key difference is that our "discrepancy from the market fundamental" only requires own type information (endowments and savings/collateral position) and the knowledge of the equilibrium price, which is a standard Walrasian assumption, while the determination of the consumption rights for each type in an adverse selection environment utilizes information on other types (see Eq. (3.2) in Bisin and Gottardi (2010) which specifies correct conjectures of what other types are doing and the related no-envy conditions in Prescott and Townsend (1984a,b)).

and sell the underlying securities and commodities, but of course these choices of prices, rights and trades must be consistent in equilibrium with the choices of all the agents. The decentralized problem achieves the same allocation as the solution to the planning problem. Note also that Arrow has individual specific prices, the price at which agent i can buy rights for consumption of agent j. We in contrast have common not type specific prices. In general, with incomplete security markets, we also need to define rights as vectors.

Interestingly, Arrow (1969) is less concerned about excludability (hence our use of the term segregation), an intrinsic part of creating the necessary markets, as he feels this has a natural counterpart in many, though not all, real world problems. Arrow (1969) is more concerned about the obvious small numbers problem. That is, his markets are not thick. However, this part is easy to remedy if we consider limit economies with a continuum of traders and positive mass of each traders type.

There is, however, an assignment problem that allows yet another extension of the commodity space and reinforces the notion of excludability. Due to some inherent non-convexities in these environments, it may be necessary to convexify the problem by allowing mixtures. These are priced so that in the decentralized problem each agent can control the probability that they are assigned to given markets and are excluded from others. What is a probability in the way an individual of a given type is treated from the individual point of view is also a fraction of the way that type is treated in the aggregate. The continuum sets of agent types with a broker dealer to pool over diverse agents is part of the technical apparatus, working for us here much the way it does for Koopmans and Beckmann (1957); Prescott and Townsend (1984a,b); Prescott and Townsend (2006).

Finally, we generalize our arguments to a large class of environments. These include that of Lorenzoni (2008), which is a cousin to our collateral example; Hart and Zingales (2013), which is a cousin to our incomplete market example; and we extend to information imperfections, a moral hazard contract economy with multiple goods and retrade in spot markets, and a Diamond-Dybvig preference shock economy with retrade in bond markets. The key, and common ingredient across environments, is a set of constraints which contain prices.

The remainder of the paper proceeds as follows. Section 2 describes the basic environment, first principles of how to remedy the externality, and structure of the two key leading examples including an exogenous incomplete markets economy and a collateral economy. Section 3 presents formal definitions of the equilibrium with rights to trade, in these two leading example economies with illustrative examples in each. We describe how to map a large variety of example economies into a generalized framework in which our market-based solution concept is applicable in Section 4. Section 5 concludes. Appendix A-D present the general model in the mixture representation, and formal proofs of the welfare and existence theorems. Additional results are contained in the online Appendices.

2 First Principles

This section features the general environment and then two leading example economics, an exogenous incomplete markets economy (Geanakoplos and Polemarchakis, 1986; Greenwald and Stiglitz, 1986) and a collateral economy (Kilenthong and Townsend, 2014b).

The general form of constraints generating pecuniary externalities for each of the two example economies is easy to write. It is denoted by

$$C^h\left(\mathbf{c}^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h, \mathbf{y}^h, \mathbf{p}\right) \geq 0, \forall h, \tag{1}$$

where h is agent type, $\mathbf{c}^h$ is consumption, $\boldsymbol{\theta}^h$ are securities, $\boldsymbol{\tau}^h$ are spot trades, $\mathbf{y}^h$ is the vector of inputs and outputs for an associated production technology, and $\mathbf{p}$ is the vector of spot prices. The key feature is the introduction of this price vector $\mathbf{p}$ into the constraints C^h. A key step to solve the resulting externality defines type h's rights to trade $\Delta_s^h(\mathbf{p})$ in the spot markets at prices p_s. Type h chooses both the amount of these rights to trade, that is, the trades at p_s, and the vector $\mathbf{p} \equiv [p_s]_{s=1}^S$ itself. There is in effect a market place exchange indexed by prices $\mathbf{p}$ where security trades will be entered into and priced at $t = 0$ and where goods will be exchanged in spot markets at state s at the same price p_s. For these rights to have meaning these exchanges must be segregated and choices of the agents must be exclusive.

The version of (1) for the generalized environment of this paper is (38) in Section 4. All of the additional environments we consider later, including those with unobserved effort and unobserved interim preference shocks, can fit into a generalized constraint, as in online Appendix I. But we can establish first principles from (1) here without more cumbersome notation. We shall come back to a class of more general constraints subsequently in Section 4. But what we can do in general to remedy pecuniary externalities is made clear here by what we can do as a remedy in the incomplete markets economy. The key feature of the environment, apart from the security structure, is that spot markets open at the two dates, and the agents face an additional constraint, in addition to their intertemporal budget constraint, where date 2 spot prices appear.

2.1 The Exogenous Incomplete Markets Economy

Consider an exogenously imposed incomplete markets economy. It is an economy with two periods $t = 0, 1$. There are S possible states of nature in the second period, $t = 1$, i.e., $s = 1, \ldots, S$, each of which occurs with probability π_s, $\sum_s \pi_s = 1$. There are 2 goods, labeled good 1 and good 2, in each date and in each state. There are H types with fractions $\alpha^h > 0$, for $h = 1, 2, ..., H$ such that $\sum_h \alpha^h = 1$. Endowment profiles are e^h_{1s} and e^h_{2s} for goods 1 and 2, where for convenience of notation $s = 0$ is the endowment at date $t = 0$. Preferences of each type h agent are represented by utility u^h.

There are J securities available for purchase or sale in the first period, $t = 0$. Let $\mathbf{D} = [D_{js}]$ be the payoff matrix of those assets in the second period $t = 1$ where D_{js} is the payoff of asset j in units of good 1 (the numeraire good) in state $s = 1, 2, \ldots, S$. Here we do not include securities paying in good 2 as there is trade in the two goods in spot markets, so these are not needed. Let θ^h_j denote the amount of the j^{th} security acquired by an agent of type h at $t = 0$ with $\boldsymbol{\theta}^h \equiv \left[\theta^h_j\right]_j$. Here a positive number denotes the purchaser or investor, and negative the issuer, the one making the promise. Let Q_j denote the price of security j with $\mathbf{Q} \equiv [Q_j]_j$. An exogenous incomplete markets assumption specifies that $\mathbf{D}$ is not full rank; that is, $J < S$. Thus agents type h cannot achieve arbitrary targets for consumption, which enter utility as $u^h\left(e^h_{1s} + \sum_j D_{js}\theta^h_j + \tau^h_{1s}, e^h_{2s} + \tau^h_{2s}\right)$, where τ^h_{1s} and τ^h_{2s} are spot trades in good 1 and 2 in state s, respectively. The u^h are strictly concave with other regularity conditions.

For this exogenous incomplete markets economy, the key set of obstacle-to-trade constraints generating the pecuniary externality are the spot budget constraints

$$C_s^h\left(\tau_{\ell s}^h, \mathbf{p}\right) \equiv \tau_{1s}^h + p_s \tau_{2s}^h = 0, \forall s, h, \tag{2}$$

where p_s is the spot market price in state s of good 2 in terms of good 1, and $\mathbf{p} \equiv [p_s]_s$ is the vector of the spot prices. This is of the form of constraint (1) mentioned earlier. The constraints here are simple spot market budget constraints, but with incomplete markets this is how prices create externalities. To be consistent with the rights to trade $\Delta_s^h(\mathbf{p})$ defined below, we keep the vector of spot prices $\mathbf{p}$ in the general constraint $C_s^h\left(\tau_{\ell s}^h, \mathbf{p}\right)$. We simplify the notation by restricting ourselves here to two periods, two goods, S states, but it is easy to generalize what we do as in Section 4. Likewise, we can easily incorporate intertemporal savings, for example the storage of good 2, $k^h \geq 0$, in Section 3.1.

2.1.1 Remedy for the Externality in the Incomplete Markets Economy

As already noted, a key step defines type h's rights to trade $\Delta_s^h(\mathbf{p})$ in the spot markets at prices p_s. Type h chooses both the amount of these rights to trade, that is, the trades at p_s, and the vector $\mathbf{p} = (p_s)_{s=1}^S$ itself. To repeat, there is in effect a market place exchange indexed by prices $\mathbf{p}$ where security trades will be entered into and priced at $t = 0$ and where goods will be exchanged in spot markets at state s at the same price p_s. For these rights to have meaning these exchanges must be segregated and choices of the agents must be exclusive.

In more detail, the quantity of rights purchased over states $s = 1, \ldots, S$ is a vector $\boldsymbol{\Delta}^h(\mathbf{p}) = \left[\Delta_s^h(\mathbf{p})\right]_{s=1}^S$. In a particular state s, $\Delta_s^h(\mathbf{p})$ is defined as the standard excess demand for the numeraire, good 1, of an agent type h in spot markets in state s. Namely, $\Delta_s^h(p_1, \ldots, p_s, \ldots, p_S) = \tau_{1s}^{h*}\left(\mathbf{e}_s^h, \boldsymbol{\theta}^h, p_s\right)$ is the solution to the state s utility maximization problem at price p_s:

$$\left(\tau_{1s}^{h*}\left(\mathbf{e}_s^h, \boldsymbol{\theta}^h, p_s\right), \tau_{2s}^{h*}\left(\mathbf{e}_s^h, \boldsymbol{\theta}^h, p_s\right)\right) = \underset{\tau_{1s}^h, \tau_{2s}^h}{\operatorname{argmax}}\, u^h\left(e_{1s}^h + \sum_{j=1}^J D_{js}\theta_j^h + \tau_{1s}^h, e_{2s}^h + \tau_{2s}^h\right) \tag{3}$$

subject to the spot-budget constraints (2). This choice of rights could be costly to buy or alternatively it could generate revenue. In particular, let $P_\Delta(\mathbf{p}, s)$ denote the market price of rights to spot trade in exchange $\mathbf{p}$ in state s, with components as desired spot prices and the vector running over all states s. Then the net cost is this per unit price times the quantity of rights $\Delta_s^h(\mathbf{p})$ just defined. Namely, $\sum_s P_\Delta(\mathbf{p}, s)\,\Delta_s^h(\mathbf{p})$ if $\mathbf{p}$ is chosen. Let $\delta^h(\mathbf{p})$ be the indicator variable which is equal to 1 for the chosen $\mathbf{p}$ and is zero otherwise. Thus the budget term will be $\sum_{\mathbf{p}} \delta^h(\mathbf{p}) \sum_s P_\Delta(\mathbf{p}, s)\,\Delta_s^h(\mathbf{p})$.

The key tie-in is that security trades $\boldsymbol{\theta}^h(\mathbf{p})$ are also tied to the choice of exchange $\mathbf{p}$. That is, let $Q_j(\mathbf{p})$ denote the price of security j traded in exchange $\mathbf{p}$. This then has a net cost in the budget $\sum_{\mathbf{p}} \delta^h(\mathbf{p}) \sum_j Q_j(\mathbf{p})\,\theta_j^h(\mathbf{p})$. Both costs of rights to trade and the tie-in to securities are subtracted from the value of endowments at $t = 0$ leaving consumption as a residual. The entire budget is the following

$$\sum_{\mathbf{p}} \delta^h(\mathbf{p}) \left[c_{10}^h + p_0 c_{20}^h + \sum_j Q_j(\mathbf{p})\,\theta_j^h(\mathbf{p}) + \sum_s P_\Delta(\mathbf{p}, s)\,\Delta_s^h(\mathbf{p}) \right] \leq e_{10}^h + p_0 e_{20}^h. \tag{4}$$

Finally, the spot prices $\mathbf{p}$ and security prices $Q_j(\mathbf{p})$ will have to be such as to attain market clearing in rights to trade:

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h \Delta_s^h(\mathbf{p}) = 0, \forall s; \mathbf{p}, \tag{5}$$

and market clearing in securities

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h \theta_j^h(\mathbf{p}) = 0, \forall j; \mathbf{p}. \tag{6}$$

Also the spot market in each state s in exchange $\mathbf{p}$ must be cleared, consistent with the agent types who have chose to trade there, validating their choice of $\mathbf{p}$.

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h \tau_{\ell s}^h(\mathbf{p}) = 0, \forall s; \mathbf{p}; \ell = 1, 2. \tag{7}$$

Note that due to the maximization of (3) subject to (2) that the chosen τ_{1s}^h at $\mathbf{p}$ that appear in (7) will be the τ_{1s}^{h*} in (3), in turn equal to the rights $\Delta_s^h(\mathbf{p})$ purchased. Finally, equations (5) can be satisfied trivially for inactive exchanges where $\delta^h(\mathbf{p}) = 0$ for all h. The full statement of the definition of equilibrium is in Section 3.

In general, spot prices p_s can be a complex mapping from pre-trade endowments and security holdings. There are few a priori restrictions on individual and aggregate excess demands. But, conceptually, for the individual this does not matter, as all she cares about are the chosen prices at which she will trade and the associated implication for rights, security, and spot trades. Finding an equilibrium is the economist's problem, not the agent's problem.

For the collateral economy though, if we assume homogeneous homothetic preferences, we can derive closed form solutions for market clearing prices and closed form solutions for demands at both the individual and aggregate level. This helps clarify the type of markets in rights we have in mind, and so we present this in the next section.

2.2 The Collateral Economy

The underlying environment for the collateral economy in terms of preferences and endowments is the same, and with the same notation, as in the incomplete markets economy. But for full generality here, and unlike the incomplete markets economy, we will consider state-contingent securities as the primitives. That is, we are dealing with an Arrow-Debreu complete securities environment. But collateral will limit the securities which emerge in equilibrium, so we have endogenous incomplete markets. Collateral will generate the externality.[5]

We consider two classes of securities[6]; (i) θ^h_{1s} - securities paying in good 1 in state s, (ii) θ^h_{2s} - securities paying in good 2 in state s. Here again a positive number denotes the

[5]A specific piece of collateral can be used to back up several contracts as long as their promises to pay are in different states. Thus there is no conflict in a given state s. This is known as *tranching*. This is distinct from the contract-specific collateralization structure (in Geanakoplos, 2003, among others), in which the collateral of a given security cannot be used as collateral for any other security. A security which would default has a known payoff structure, so we may as well start with that in the first place; there is no loss of generality in restricting attention to securities without default. But the possibility of default does restrict securities, and collateral constraints can be binding. Further, issuing securities that do default requires no less collateral than (an equivalent set of) securities that do not. See Kilenthong and Townsend (2014b).

[6]Actually with spot markets we need securities θ^h_{1s} paying in the numeraire only. We proceed here in more generality as what we do will not require active spot markets. As shown in Kilenthong and Townsend (2014b), spot markets are redundant when all types of state contingent contracts are available ex-ante. In other words, agents do not really need to trade in spot markets even though they may well do so. But promises in ex ante markets still need to be backed by collateral in escrow. The general environment below in Section 4 allows security prices to enter into collateral constraints.

purchaser or investor, and negative the issuer, the one making the promise. When negative, each of the state-contingent securities must be backed, so that the promise is honored. Only good 2 can be stored, so only good 2 can serve as backing.

Let $k^h \in \mathbb{R}_+$ denote the collateral holding (equivalent to the holding of good 2) of an agent type h at the end of period $t = 0$ to be carried to period $t = 1$. Note that this collateral allocation does not need to be equal to the initial endowment of good 2, e^h_{20}. In particular, since good 2 can be exchanged or acquired in the contracting period at date $t = 0$, k^h will be equal to the net position in the collateral good after trading in period $t = 0$. The collateral good as legal collateral backing claims is assumed to be fully registered and kept in escrow, i.e., cannot be taken away or stolen, neither by borrowers nor lenders. However, the holding of good 2 can also include normal saving. The storage technology of good 2, whether in collateral or normal savings, is linear but with a potentially random return. In some applications, it is natural to treat the returns as a constant and focus on how collateral interacts with intertemporal trade. In other applications, the risk is in the collateral itself. Each unit of good 2 stored will become $R_s > 0$ units of good 2 in state $s = 1, \cdots, S$. Specifically, storing I units of good 2 at date $t = 0$ will deliver $R_s I$ units of good 2 in state s at $t = 1$.

A collateral constraint for each state s states that the net value of all assets, including collateral good and securities, must be non-negative. If θ^h_{1s} and θ^h_{2s} were negative, as promises, we would write the constraint as $p_s R_s k^h \geq -\theta^h_{1s} - p_s \theta^h_{2s}$. That is, there is sufficient collateral in value in state s to honor the value of all such promises. Note that all promises are converted to units of good 1 using the spot market price of the collateral good p_s. For example, a promise to deliver units of good 1 can be honored by converting good 2 in collateral to good 1 at prices p_s. More generally a type h may hold investments $\theta^h_{\ell s}$, positive, and these can also be used in value to honor promises. Thus, in sum, the obstacle-to-trade constraint, generating the pecuniary externality, can be written as

$$p_s R_s k^h + \theta^h_{1s} + p_s \theta^h_{2s} \quad \geq \quad 0, \ \forall s, h, \tag{8}$$

or in the notation we introduced at the outset, (1),

$$C^h_s \left(\theta^h_{\ell s}, y^h_{2s}, p_s\right) \equiv p_s y^h_{2s} + \theta^h_{1s} + p_s \theta^h_{2s} \geq 0, \forall s, h, \tag{9}$$

where y^h_{2s} is storage for the general notation.

In this collateral economy we can establish that if any type h collateral constraint is binding, the allocation is not constrained optimal. There is too much saving, hence the price of the collateral good in each state s is too low, too much good 2 around, and the price at $t = 0$ is too high, too much of good 2 is saved. The externality is associated with too much storage. Agents do not internalize the externalities that making promises implies storage which in turn impacts future spot prices. See the details in Kilenthong and Townsend (2014b) and online Appendix D.

2.2.1 Remedy for the Externality in the Collateral Economy

With identical homothetic preferences, the aggregate ratio of good 1 to good 2 in state s is the market fundamental in state s, z_s, that determines price $p_s = p(z_s)$; that is,

$$z_s = \frac{\sum_h \alpha^h e^h_{1s}}{R_s K + \sum_h \alpha^h e^h_{2s}}, \tag{10}$$

where $K = \sum_h \alpha^h k^h$ is the aggregate (endogenous) saving including collateral backing promises. The point is that ratios of the aggregate are enough to pin down equilibrium prices p_s. We can easily go back and forth, referring to price $p(z_s)$ or the underlying fundamental ratio z_s. The latter has analytic advantages.

Specifically, with common CRRA utility functions $u(c_1, c_2) = \frac{c_1^{1-\gamma}-1}{1-\gamma} + \frac{c_2^{1-\gamma}-1}{1-\gamma}$, the excess demand function for good 1, the numeraire, with the market fundamental z_s is given by $\tau^{h*}_1(z_s) = \left(\frac{z_s^{\gamma-1}}{1+z_s^{\gamma-1}}\right)\Delta^h_s(z_s)$, where $\Delta^h_s(z_s)$ is the discrepancy of a type h from the fundamental z_s:

$$\Delta^h_s(z_s) = \left(e^h_{2s} + R_s k^h\right)\left(z_s - \frac{e^h_{1s}}{e^h_{2s} + R_s k^h}\right). \tag{11}$$

We will as earlier internalize the externality by having agents choose the price $p(z_s)$, equivalently the fundamental ratio z_s, at which they want to trade and the corresponding excess demand $\tau^{h*}_1(z_s)$, equivalently the discrepancy $\Delta^h_s(z_s)$. Comparing $\tau^{h*}_1(z_s)$ and $\Delta^h_s(z_s)$, the additional terms in the excess demand $\left(\frac{z_s^{\gamma-1}}{1+z_s^{\gamma-1}}\right)$ depends on z_s only. Thus one can go back and forth between discrepancy $\Delta^h_s(z_s)$ and excess demand $\tau^h_s(z_s)$ simply by redefining the units traded in each state s. To repeat, in the markets for rights one can price either, that is, either $\Delta_s^h(z_s)$ as we will do here, or $\tau_1^{h*}(z_s)$ as we did in the previous section. Nothing fundamental changes. The discrepancy reflects the good 2 weighed gap between the market ratio z_s and the pre-trade ratio of type h. In addition summing over h, weighted by the mass of type h, α^h, yields the following market clearing condition in state s when the fundamental is z_s,

$$\sum_h \alpha^h \Delta_s^h = z_s \sum_h \alpha^h \left(e_{2s}^h + R_s k^h\right) - \sum_h \alpha^h e_{1s}^h = 0. \tag{12}$$

Intuitively, not everyone can be above or below the average, so the average of the discrepancies must be zero. In this sense $\Delta_s^h(z_s)$, just as with excess demand, is the contribution of type h to the equilibrium price. Indeed, equation (12) is literally aggregate excess demand if we scale by $\left(\frac{z_s^{\gamma-1}}{1+z_s^{\gamma-1}}\right)$.

The analytics of homotheticity help us to decompose aggregate demand determining prices into its type h components, to help us think through the fundamental economics of the problem. Note that an advantage of the discrepancy $\Delta_s^h(z_s)$ is its independence from the utility function parameters. To internalize the externality, agents who bring in too much of good 2 relative to the market fundamental would have to pay for rights to trade, and vice versa, be compensated for having a relative large amount of good 1. For example, if $\Delta_s^h(z_s) > 0$, then from (11), $z_s > \frac{e_{1s}^h}{e_{2s}^h + R_s k^h}$ and type h holds a relatively low amount of good 1 and an abundant amount of good 2, relative to z_s. As a result, that type h would need to pay for the right to trade or unwind in this market z_s. This makes intuitive sense since the problem, the inefficiency as noted earlier, is oversaving. Conversely, when $\Delta_s^h(z_s) < 0$, an agent type h has a relatively high amount of good 1 and scarce amount of good 2, relative to z_s. With oversaving in the aggregate, this type will be compensated for this behavior, buying good 2. Note that if type h's pretrade endowment were exactly equal to the market fundamental, $\frac{e_{1s}^h}{e_{2s}^h + R_s k^h} = z_s$, then $\Delta_s^h(z_s) = 0$. That agent type is not putting pressure on the market price, one way or the other. But typically, with heterogeneity and active trade, an agent type h will be on one side or the other of the market fundamental, buying or selling the collateral good 2 for good 1. Of course it takes at least two sides to open an active market.

We continue to assume here that each agent must choose one but only one fundamental spot market in each state s, that is, one price at which collateral will be unwound. More formally, let an indicator function $\delta^h(z_s) = 0$ or $\delta^h(z_s) = 1$, denote an agent type h's discrete choice of spot and security market z_s in state s, and do this for each s, $s = 1, 2, \ldots, S$. Unlike the incomplete markets economy of the previous section, the complete markets security structure here allows us to write $\delta^h(z_s)$ state by state. Previously we wrote $\delta^h(\mathbf{p})$ over the entire vector $\mathbf{p}$. Nevertheless, with vector $\mathbf{z} = (z_s)_{s=1}^S$, we denote the state contingent choices here as a vector $\delta^h(\mathbf{z}) \equiv \left[\delta^h(z_s)\right]_{s=1}^S$, again with the particular z_s in each s as specified. In equilibrium not all markets z_s will be active, but some will be.

Now let $P_\Delta(z_s, s)$ be the unit price of rights to trade at chosen spot-market price $p_s = p(z_s)$. The fee for the rights to trade in a security exchange z_s is given by that price, $P_\Delta(z_s, s)$, times the quantity of the discrepancy, $\Delta_s^h(z_s)$, namely the total expenditures is $P_\Delta(z_s, s)\,\Delta_s^h(z_s)$ (or revenue if negative). Thus in total, summing over all states and taking into account the chosen fundamental z_s, we will have in the budget constraint the expense term $\sum_{\mathbf{z}} \delta^h(\mathbf{z}) \sum_s P_\Delta(z_s, s)\,\Delta_s^h(z_s)$.

These choices of $\mathbf{z}$ are jointly bundled with securities $\theta_{\ell s}^h$. We could write this as $\theta_{\ell s}^h(\mathbf{z})$ consistent with the incomplete markets economy, but here we can write this more simply as state contingent, $\theta_{\ell s}^h(z_s)$. These securities are posted at prices $Q_\ell(z_s, s)$. Thus in the $t = 0$ budget constraint there is an expense (or revenue), $\sum_{\mathbf{z}} \delta^h(\mathbf{z}) \sum_s \sum_\ell Q_\ell(z_s, s)\,\theta_{\ell s}^h(z_s)$. Finally, the pre-committed trades $\tau_{1s}^h(z_s)$ and $\tau_{2s}^h(z_s)$ must be such that the agent h is on her budget line at prices $p(z_s)$:

$$\sum_{\mathbf{z}} \delta^h(\mathbf{z}) \left[\tau_{1s}^h(z_s) + p(z_s)\,\tau_{2s}^h(z_s)\right] = 0, \forall s, \tag{13}$$

where again, as before, the spot trades are consistent with the discrepancies and rights to trade.

3 Formal Statement of Equilibrium with Rights to Trade, with Illustrative Examples

A competitive equilibrium with segregated exchanges is similar to the standard definition except that the objective function and constraints are premultiplied by discrete choice, the budget is augmented by expenditures on rights, and there is an additional clearing equation for these rights. Though this may be obvious from the previous section, we summarize the discussion here.

3.1 Definition of Equilibrium with Rights to Trade

Let $\mathbf{x}^h(\mathbf{p}) = \left(\mathbf{c}_0^h, k^h, \delta^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p}), \boldsymbol{\tau}^h(\mathbf{p}), \boldsymbol{\Delta}^h(\mathbf{p})\right)$ denote a typical bundle or allocation for an agent type h, where again $\boldsymbol{\Delta}^h(\mathbf{p}) \equiv \left[\Delta_s^h(p_s)\right]_s$. If $\delta^h(\mathbf{p}) = 0$, then the rest of the $\mathbf{p}$ contingent choices need not be specified, as agent h is choosing not to trade at $\mathbf{p}$.

Definition 1. A competitive equilibrium with segregated exchanges indexed by $\mathbf{p}$ is a specification of allocation $\left[\mathbf{x}^h(\mathbf{p})\right]_{h,\mathbf{p}} \equiv \left[\mathbf{c}_0^h, k^h, \delta^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p}), \boldsymbol{\tau}^h(\mathbf{p}), \boldsymbol{\Delta}^h(\mathbf{p})\right]_{h,\mathbf{p}}$ and prices $(p_0, \mathbf{Q}, \mathbf{p}, \mathbf{P}_\Delta)$ such that

(i) for any agent type h as a price taker, $\left[\mathbf{x}^h(\mathbf{p})\right]_{\mathbf{p}}$ solves

$$\max_{\left[x^h(\mathbf{p})\right]_{\mathbf{p}}} \sum_{\mathbf{p}} \delta^h(\mathbf{p}) \left[u\left(c_{10}^h, c_{20}^h\right) + \sum_s \pi_s u\left(e_{1s}^h + \sum_j D_{js}\theta_j^h(\mathbf{p}) + \tau_{1s}^h(\mathbf{p}), e_{2s}^h + R_s k^h + \tau_{2s}^h(\mathbf{p}) \right)\right]$$

subject to the budget constraints in the first period

$$\sum_{\mathbf{p}} \delta^h(\mathbf{p}) \left[c_{10}^h + p_0\left(c_{20}^h + k^h\right) + \sum_j Q_j(\mathbf{p})\,\theta_j^h(\mathbf{p}) + \sum_s P_\Delta(\mathbf{p}, s)\,\Delta_s^h(\mathbf{p}) \right] \leq e_{10}^h + p_0 e_{20}^h,$$

and the spot-budget constraint in state s

$$\sum_{\mathbf{p}} \delta^h(\mathbf{p}) \left[\tau_{1s}^h(\mathbf{p}) + p_s \tau_{2s}^h(\mathbf{p})\right] = 0, \forall s,$$

(ii) markets clear for good ℓ in $t = 0$, for securities j paying good 1, for good ℓ in state s, and for rights to trade in exchange $\mathbf{p}$ for state s, respectively,

$$\sum_h \alpha^h c_{10}^h = \sum_h \alpha^h e_{10}^h,$$

$$\sum_h \alpha^h \left(c_{20}^h + k^h\right) = \sum_h \alpha^h e_{20}^h,$$

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h \theta_j^h(\mathbf{p}) = 0, \forall j; \mathbf{p},$$

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h \tau_{\ell s}^h(\mathbf{p}) = 0, \forall s; \mathbf{p}; \ell = 1, 2,$$

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h \Delta_s^h(\mathbf{p}) = 0, \forall s; \mathbf{p}.$$

For the collateral economy, there is of course the additional constraints (9). The collateral economy is also special in that in addition to these collateral constraints, the security structure is complete, that is, the securities, θ_{1s}^{h} and θ_{2s}^{h}, are the Arrow-Debrue securities, the selection $\delta^{h}(\mathbf{z})$ and indexation of securities, spot trades $\tau_{\ell s}^{h}(z_s)$, the rights $\Delta_{s}^{h}(z_s)$ and prices $p(z_s)$ can all be written in terms of the fundamental ratio z_s, and everything is state contingent (no need for vectors). See its formal definition in the mixture representation in Section 3.4 and without lotteries in online Appendix E.

3.2 Illustrative Examples for the Collateral Economy

3.2.1 Public Finance Interpretation

The budget constraint with the prices of the rights to trade for the collateral has a public finance interpretation, as if we were to try to implement the optimum solution by taxes and subsidies.

Specifically, substituting $\Delta_{s}^{h}(z_s) = z_s\left(e_{2s}^{h} + R_s k^{h}\right) - e_{1s}^{h}$ into the following budget constraint for an agent type h:

$$\sum_{\mathbf{z}} \delta^{h}(\mathbf{z}) \left\{ c_{10}^{h} - e_{10}^{h} + p_0 \left[c_{20}^{h} + k^{h} - e_{20}^{h} \right] + \sum_{s} \sum_{\ell} Q_{\ell}(z_s, s)\, \theta_{\ell s}^{h}(z_s) + \sum_{s} P_{\Delta}(z_s, s)\, \Delta_{s}^{h}(z_s) \right\} \leq 0,$$

gives the following interpretable budget constraint:

$$\sum_{\mathbf{z}} \delta^{h}(\mathbf{z}) \left\{ c_{10}^{h} - e_{10}^{h} + p_0 \left[c_{20}^{h} + k^{h} - e_{20}^{h} \right] + \sum_{s} \sum_{\ell} Q_{\ell}(z_s, s)\, \theta_{\ell s}^{h}(z_s) \right\} \leq$$
$$\sum_{\mathbf{z}} \delta^{h}(\mathbf{z}) \left\{ - \left[\sum_{s} P_{\Delta}(z_s, s)\, z_s R_s \right] k^{h} - \sum_{s} \left[P_{\Delta}(z_s, s)\, z_s \right] e_{2s}^{h} - \sum_{s} \left[-P_{\Delta}(z_s, s) \right] e_{1s}^{h} \right\}.$$

We can now see that we need to have three types of taxes/subsidies, (i) saving/collateral tax of $\sum_s P_{\Delta}(z_s, s)\, z_s R_s$ per unit of saving/collateral, k^{h}, (ii) state-contingent collateral good endowment tax of $P_{\Delta}(z_s, s)\, z_s$ per unit of collateral good 2 endowment in state s, e_{2s}^{h}, and (iii) state-contingent subsidy, negative tax $-P_{\Delta}(z_s, s)$ per unit of consumption good endowment of good 1 in state s, e_{1s}^{h}. Note that it is not sufficient to tax/subsidize only saving/collateral. The tax/subsidy rate on endowments also depends on the security exchange z_s chosen. That is, the security exchange z_s itself is a choice as far as the household is concerned, so these are not lump sum endowment taxes.[7]

But again we do not need the taxes. We let the markets decide. Markets determine prices, and prices determine allocations.

3.2.2 Example Economies

We present a series of examples. The first illustrates the basic mechanics of rights to trade we have featured so far in the simplest possible setting without uncertainty (in which in equilibrium there are no securities beyond simple saving). The second features two states of the world in which insurance contracts are actively traded in both the equilibrium with and without the externalities.

Environment 1 (Intertemporal Smoothing)**.** There are two periods, $t = 0, 1$, and a single state, $S = 1$ in period $t = 1$. So this is a pure intertemporal economy. We thus make the point that the problem and its remedy has nothing to do with uncertainty. In particular, our rights are not trades on financial options. Indeed, in this example economy no securities will be traded, in equilibrium, and in this way we focus on the market for rights to trade in spot markets, only. Henceforth we drop all subscript s from the notation.

There are two types of agents, $H = 2$, both of which have an identical constant relative risk aversion (CRRA) utility function[8]

$$u(c_1, c_2) = -\frac{1}{c_1} - \frac{1}{c_2}. \tag{14}$$

Each type h consists of $\frac{1}{2}$ fraction of the population, i.e., $\alpha^h = \frac{1}{2}$. In addition, the common discount factor is $\beta = 1$. The storage technology is given by $R = 1$. The endowment profiles of the agents are shown in Table 1 below. Note that an agent type 1 is well endowed with both goods in period $t = 0$ relative to $t = 1$, and vice versa for type 2. The first best allocation has both agents consuming 2 units of each good in every period. The first-best allocation with full commitment and hence no collateral constraints thus suggests the obvious, that agent

[7] This is like looking up marginal rates in a big tax book and settling on which page (or pages) to use, indexed by the exchanges z_s that the agent chooses.

[8] Here we set the coefficient of relative risk aversion $\gamma = 2$, and drop the irrelevant constant term.

type 2 would like to move resources backwards in time from $t = 1$ to $t = 0$, i.e., borrow, and therefore will be constrained in the competitive collateral equilibrium. Borrowing requires collateral, and agent type 2 is short of this as well. The equilibrium with the externality present, and also the one with the externality corrected, will have agent type 2 borrowing nothing and only trading in spot markets. Agent type 1 will be saving on its own to smooth consumption over time.

We summarize the equilibrium allocation in Table 1 featuring collateral k^h and consumption $c_{\ell s}^h$. See online Appendix H.1 for the derivation of the competitive equilibrium with the externality.

Table 1: Endowment profiles of the agents.

	endowments				equilibrium with the externality (ex)					equilibrium with rights to trade (op)				
	e_{10}^h	e_{20}^h	e_{11}^h	e_{21}^h	k^h	c_{10}^h	c_{20}^h	c_{11}^h	c_{21}^h	k^h	c_{10}^h	c_{20}^h	c_{11}^h	c_{21}^h
$h = 1$	3	3	1	1	1.360	2.690	1.776	1.325	1.776	1.175	2.607	1.841	1.297	1.678
$h = 2$	1	1	3	3	0	1.310	0.865	2.675	3.584	0	1.393	0.984	2.703	3.497

There is no loss of generality to consider a solution with no security trading, i.e., $\theta_{\ell s}^h = 0$ for all h and for all ℓ. Agents do however actively trade in spot markets. With the externality (denoted "ex"), the price of good 2 in period $t = 0$ is $p_0^{ex} = \left(\frac{4}{4-k^{ex}}\right)^2 = 2.2948$, and the market fundamental in period $t = 1$ is $z^{ex} = \frac{4}{4+k^{ex}} = 0.7463$, which implies that the spot price is $p\left(z^{ex}\right) = 0.5570$.

We will now turn to a corresponding competitive equilibrium with rights to trade in segregated exchanges (without the externality, the one with rights to trade that we have featured). There is *one and only one active spot market at* $t = 1$, $z^{op} = 0.7729$ ("op" stands for optimality; we have not yet proved the first welfare theorem for markets with rights to trade, but it will apply), even though all spot markets are available in principle for trade. That is, in equilibrium, both types optimally choose to trade in the same spot market with specified market fundamental, $z^{op} = 0.7729$. Note that as anticipated there is less saving with the externality corrected, so the spot price of good 2 is lower at $t = 0$ and is higher at $t = 1$ relative to the equilibrium with externalities present.

Table 2 presents equilibrium prices/fees of rights to trade in spot markets, that is $P_\Delta(z)$ not only for z^{op} but also other, different market fundamental levels z. We are here defining

$P_\Delta(z) = P_\Delta(z_s, s)$ where there are no states so we have dropped the s notation. Note again that the prices/fees of non-active spot markets are available, but at such prices agents do not want to trade in them.

Table 2: Equilibrium prices of rights to trade in spot markets $P_\Delta(z)$. Bold numbers are equilibrium prices for actively traded spot markets.

	$z = 0.7479$	$\mathbf{z = 0.7729}$	$z = 0.7979$
$P_\Delta(z)$	0.4639	**0.5375**	0.6118

An agent type 1 is coming in with good 2 in storage, and therefore his discrepancy is positive. Type 1 pays for right to trade. This makes sense as agent type 1 is doing the saving in good 2 and there is too much saving in the (ex) equilibrium. On the other hand, an agent type 2's discrepancy is negative. Thus, with a positive equilibrium fee $P_\Delta(z^{op}) = 0.5375$, an agent type 2 will be paid for her willingness to choose that market. Agent type 2 is facing a higher price of the good 2, that will be purchased. But there is compensation. In particular, a constrained agent ($h = 2$) with $\Delta^2(z^{op}) = -0.6813$, is receiving $-P_\Delta(z^{op})\Delta^2(z^{op}) = 0.3662$ in period $t = 0$ for being in the spot market $z^{op} = 0.7729$. Graphically, this shifts her budget line outward at $t = 0$ by $T = 0.3662$, hence in the direction of being less constrained.[9]

The next example illustrates an economy with uncertainty where collateralized securities, θ^h_{1s}, are actively traded. All agents are constrained, but at different states. In particular, an agent will be binding in a state where her endowment is large, as it is for such states that she would make a promise to pay, and promises must be backed by collateral.

Environment 2 (State Contingent Securities)**.** The economy in this example is similar to the one in Environment 1 with two periods, but there are two states, $S = 2$ at $t = 1$. The

[9] Trading in rights to trade generates a redistribution of wealth and welfare in general equilibrium. The expected utility of an agent type 1 and type 2 in this competitive equilibrium with segregated exchanges (without the externality) are $U^1_{op} = -2.2936, U^2_{op} = -2.3905$, respectively. The expected utility of an agent type 1 and type 2 in the competitive collateral equilibrium allocation (with the externality) are $U^1_{ex} = -2.2527$ and $U^2_{ex} = -2.5724$, respectively. Thus if nothing else were done, internalizing the externality would be beneficial to an agent type 2 (constrained agent) but harmful for an agent type 1. To induce welfare gains for all of agents, there must be lump sum transfers, as in the second welfare theorem, which we also prove below.

endowment profiles are presented in Table 3. Note, unlike the first example, that the agents are identical in endowments at $t = 0$. But agent type 1 has relatively more of both goods in state $s = 1$ than in state $s = 2$, and vice versa for agent type 2.

Table 3: Endowment profiles of the agents.

	e^h_{10}	e^h_{20}	e^h_{11}	e^h_{21}	e^h_{12}	e^h_{22}
$h = 1$	2	2	3	3	1	1
$h = 2$	2	2	1	1	3	3

We will now solve for a competitive equilibrium with the externality. The detailed derivation is again omitted and presented in online Appendix H.2. An agent type $h = 1$ issues $\theta^1_{11} = -0.3042$ units of collateralized security paying good 1 at $s = 1$, that is, promises to pay at $s = 1$, and invests the same amount $\theta^1_{12} = 0.3042$, to be paid at $s = 2$. Vice versa for an agent $h = 2$. We now turn to the competitive equilibrium with rights to trade. Each type of agent holds the same amount of collateral good $k^{op} = 0.4200 < 0.4603$, less than the one in competitive equilibrium with the externality, as anticipated. Collateralized securities are of the same sign but overall payments are less $\theta^1_{11} = -\theta^1_{12} = -0.2872 = -\theta^2_{11} = \theta^2_{12}$. That is, there is less volume in the securities markets relative to the equilibrium with the externality. This is again because the agents save less, consistent with issuing fewer securities, hence less collateral.

The collateral economy has been described as if there were active trade in spot markets. But here, with a complete set of state contingent securities traded ex ante, there is an equivalent solution in which all trade takes place in the ex ante security markets (as usual). Promises to deliver, negative positions, still require collateral, however. A collateral constraint can now be written with ratio of security prices, generating exactly the same pecuniary externality. The two formulations are equivalent. See online Appendix I for details.

3.3 Illustrative Example for the Exogenous Incomplete Markets Economy

The definition of competitive equilibrium in an incomplete markets economy is standard. Again see online Appendix A.1. We also write out in online Appendix B as noted earlier the definition of equilibrium in the notation of the collateral economy environment.

With incomplete security markets, each security exchange must naturally deal with S spot markets as a bundle since a given security exchange at $t = 0$ has implications in general for most if not all spot prices. A bond pays off in all states, for example. If the security markets were complete, these indirect price effects would be canceling each other out, and as a result, the competitive equilibrium with exogenous security markets would be efficient as expected. This statement is formally proved in a proposition in online Appendix A.2. These non-cancelling price effects are featured there.

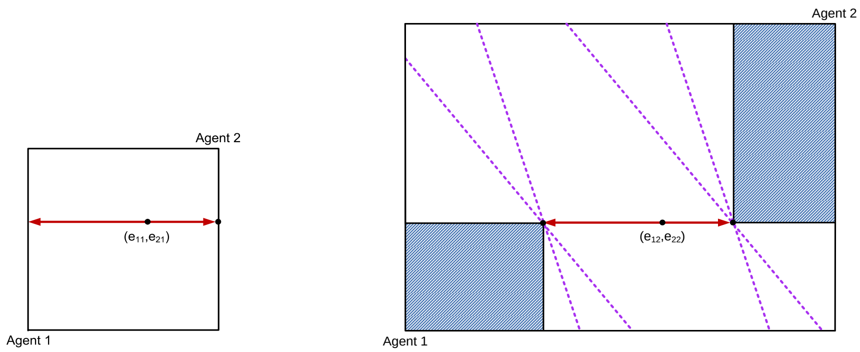
As in the collateral economy, markets for the rights to trade can remove the externality, as we prove formally below. Of course, one might wonder if our method solves the externality problem by simply completing the markets? By allowing agents to choose markets with pre-specified spot prices in each state s, we effectively create state-contingent transfers of wealth at least to some degree. But is it enough to achieve the first best allocation? The answer is generally, no. Exogenous incomplete markets and the positivity of spot prices still restrict how much wealth transfers we can make in each state. Technically, the feasible set with incomplete markets and rights to trade is generically a strict subset of the feasible set with the complete markets.

See Figure 1 for an illustrative example. This example assumes the stereotypical debt contract, a bond that pays the same amount of good 1 in each two states. However, in state $s = 2$, there is more of good 1 and good 2 overall. Then, no matter what the price ratio p_s in state $s = 2$, certain regions cannot be reached. The main point is that the scarcity in state $s = 1$ can affect the feasibility in state $s = 2$ because the markets are incomplete.

3.3.1 An Example: Saving-Borrowing Economy with Segregated Exchanges

The following example presents an incomplete markets economy with saving and borrowing only.

Environment 3. Consider an economy with two periods, $t = 0, 1$. There are two states $s = 1, 2$ in the second period. Let the probability of state s is $\pi_s = \frac{1}{2}$. For simplicity, there is only one good in the first period $t = 0$, while there are two goods $\ell = 1, 2$ in the second period $t = 1$. Each unit of storage of the single good in the first period becomes one unit



(a) A feasible set in state $s = 1$.

(b) A feasible set in state $s = 2$. The shaded areas are not feasible.

Figure 1: Feasible Sets in state $s = 1$ and $s = 2$ when markets are incomplete. The only available security pays the same amount of good 1 in both states.

of good 2 in the second period regardless of the state (no shocks on the return to storage). Notationally, k units of storage in $t = 0$ gives k units of good 2 in every state in the second period.

There are two types of agents, $H = 2$, both of which have an identical logarithmic utility function

$$u^h(c) = \ln c, \tag{15}$$

which is homothetic. As a result, the equilibrium spot price p_s is determined by the ratio of the commodity aggregates, as in the collateral economy. In particular, with the log utility, $p_s = z_s$. Note that the externality exists in this economy due to the interaction between the incompleteness of the markets and storage, as first period decisions impact the second period price. Each type consists of $\frac{1}{2}$ fraction of the population, i.e. $\alpha^h = \frac{1}{2}$. In addition, the discount factor is $\beta = 1$.

The endowment profiles of the agents are shown in Table 4 below. Note that all risk is idiosyncratic, with type 1 relatively well endowed in state $s = 1$ and vice versa for type 2. Note also that the symmetry of the endowments and the homogeneity of the preferences imply that an equilibrium allocation is symmetric. In addition, the endowment is structured in such a way that both types would like to save ($k^h > 0$).

With symmetry, the first best allocation ("fb") with state contingent transfers has both

Table 4: Endowment profiles of the agents.

	e_{10}^h	e_{11}^h	e_{21}^h	e_{12}^h	e_{22}^h
$h=1$	10	5	5	1	1
$h=2$	10	1	1	5	5

agents save $k_0^{fb} = 3.5$, which implies that the equilibrium spot price of good 2 $p_s^{fb} = 0.4615$ in all states. On the other hand, with the externality, each agent type saves more $k^{ex} = 4.3077$ to try to cover some of the idiosyncratic risk, which leads to a lower equilibrium spot price $p_s^{ex} = 0.4105$ in all states.

Similar to the collateral economy, with identical logarithm utility functions, the rights to trade in exchange $\mathbf{p}$ in state s is defined by

$$\Delta_s(\mathbf{p}) = p_s\left(e_{2s}^h + k^h\right) - e_{1s}^h, \tag{16}$$

which is the standard excess demand of type h for the numeraire. The equilibrium allocation with rights to trade in segregated exchanges has only *one active spot market* $\mathbf{p}^{op}$ with $p_s^{op} = 0.4272$ for all $s = 1, 2$. The equilibrium savings here is lower than the one in competitive equilibrium with the externality, i.e., $k^{op} = 4.0233$ (but still higher than in the first best). Table 5 presents equilibrium prices/fees of rights to trade in exchange vector $\mathbf{p} = (p_1, p_2)$ in each state $s = 1, 2$, the $P_\Delta(\mathbf{p}, s)$ with argument ranging over $\mathbf{p}$ and s that is, including over inactive exchanges. Note that the prices of the rights to trade with the spot price p_s in different exchanges are clearly different, i.e., $P_\Delta((p_1, p_2), 1) \neq P_\Delta((p_1, \tilde{p}_2), 1)$ when $p_2 \neq \tilde{p}_2$. Note that here the vector is different in the second component, yet this makes the rights price for trading in the first state different.

In this equilibrium, each agent type buys/sells the rights to trade in an exchange $\mathbf{p}^{op} = (0.42715, 0.42715)$. Due to symmetry, an agent type $h = 1$ sells the rights $\Delta^1(\mathbf{p}^{op}, 1) = -1.1457$ in state $s = 1$, the numeraire good 1 and hence buys good 2 in $s = 1$. Agent type $h = 1$ buys the same amount of good 1 in state $s = 2$, $\Delta^1(\mathbf{p}^{op}, 2) = 1.1457$, and hence agent type 1 sells good 2 at $s = 2$. This is crucial as savings of type 1 is motivated by the shortfall of type 1's endowment in state $s = 2$. That is, this is where the exposure to idiosyncratic risk for agent type 1 is doing damage, bringing too much good 2 to the second period, creating the externality. The markets for rights to trade in good 1 can remove the

Table 5: The equilibrium price of the rights to trade in exchange $\mathbf{p} = (p_1, p_2)$ in each state s, $P_\Delta(\mathbf{p}, s)$.

$\mathbf{p}$		$P_\Delta(\mathbf{p}, 1)$	$P_\Delta(\mathbf{p}, 2)$
p_1	p_2		
0.41609	0.41609	0.07456	0.07456
0.41609	0.42715	0.07807	0.09456
0.42715	0.41609	0.09455	0.07806
0.42715	**0.42715**	**0.13342**	**0.13342**
0.42715	0.46154	0.17094	0.23213
0.46154	0.42715	0.23213	0.17094

externality through its marginal impact on the decision to save of each agent type. In effect, each pays a "tax" when selling good 2 and buying good 1 in the state which motivated the saving. The situation is reversed for agent type 2 in terms of the ordering over goods and states, but the same in terms of saving. In total, the net trade in the rights to trade will be zero in net value for both agent types at $t = 0$. What agent 1 buys agent 2 sells and vice versa, and each trade has the same value. That is, each agent does actively trade rights but with no implication for wealth at $t = 0$. The key is the marginal impact of each rights to trade on saving decisions.

The markets for rights to trade do not complete the securities markets, as evident in the result that the equilibrium outcome is identical to the constrained optimal one and is not the first best allocation. There remains considerable variation of consumption over states for each type. Still, the markets for rights will result in a constrained Pareto optimal allocation, as shown in online Appendix G. Here the target constrained optimal allocation can be derived simply by maximizing the expected utility of the $t = 0$ representative consumer, exploiting the symmetry, subject to spot market constraints with price p_s replaced by the appropriate clearing ratio of commodities. The solution to this planning problem is confirmed to be the solution to the equilibrium with rights.

3.4 Competitive Equilibrium with Segregated Exchanges in The Mixture Representation

The following example breaks new ground and presents an economy where it is possible to assign agents to different exchanges and to have multiple segregated exchanges. We return to the collateral economy for simplicity of notation and ideas.

Environment 4 (Heterogeneous Borrowers and the Role of Mixtures)**.** Apart from more heterogeneity in agent types, all other aspects of the environment are as in Environment 1-where agent type 1 was lender/saver. But now there are three types of agents, with two borrower types 2, 3. Each type consists of $\frac{1}{3}$ fraction of the population, i.e. $\alpha^h = \frac{1}{3}$. The endowment profiles are given in Table 6, below. As in Environment 1, there is no uncertainty.

Table 6: Endowment profiles of the agents.

Type of Agents	e^h_{10}	e^h_{20}	e^h_{11}	e^h_{21}
$h = 1$	4.26	11.5	0.5	0.5
$h = 2$	3.92	0.5	7	5
$h = 3$	4.32	0.5	5	7

Yet interestingly, there are now *two active spot markets*, $z = 0.6113$ and $z = 0.8132$ in the competitive equilibrium with segregated exchanges.[10] The spot market $z = 0.6113$ consists of some fraction of agents type 1 (19.69 percent), and all of agents type 3 (that is, all of the constrained agents of a certain type). On the other hand, the spot market $z = 0.8132$ consists of some residual fraction of agents type 1 (80.31 percent) and all of agents type 2 (all of the other constrained type). We use the term mixtures to refer to the fact that agent 1 is allocated to two active markets in some nontrivial proportions.

Equilibrium fees of rights to trade in spot markets, including the fees of inactive spot markets are summarized in Table 7 below.[11]

[10] Note that the active markets are discretely separated from one another, i.e., the separation is not numerical but real. We have computed equilibrium for various refinements of the underlying space of zs and one gets discrete separation, as in Table 7, where there are two inactive exchanges between the active zs. We conjecture there are as many active markets as there are constrained types but have been unable to prove.

[11] Of course, ex-post we could have shut the inactive ones down, but we could not know which a priori.

Table 7: Equilibrium fees of rights to trade in spot markets. The bold numbers are the equilibrium prices of actively traded exchanges.

	$z = 0.6088$	$\mathbf{z = 0.6113}$	$z = 0.6138$	$z = 0.8088$	$\mathbf{z = 0.8132}$	$z = 0.8138$
$P_{\Delta}(z)$	0.9119	**0.9348**	0.9589	2.2339	**2.2537**	2.2564

It is socially optimal to compensate constrained agents with positive transfers at period $t = 0$, to try to move back toward the first best, i.e., alleviate borrowing constraints. In this example the number of active segregated spot markets is equal to the number of constrained types, to allow this to happen.

Specifically, in the competitive equilibrium with segregated exchanges (without the externality), the discrepancy from the market fundamental of both constrained types are negative, i.e., $\Delta^2 = -2.9340$ and $\Delta^3 = -0.7209$. With a positive equilibrium price of the discrepancy in each active market, agents type 2 and agents type 3 are paid for the chosen rights to trade at fees $P_{\Delta}(z)\,\Delta^h(z)$. Agents type 1 buy a lottery which is actuarially fair; fees are paid in proportion to the relative number of its type assigned to each exchange. Agents type 1 would like to buy into the higher spot market, $z = 0.8132$ in this case, with certainty, where good 2 is more valuable because with her endogenous saving she would benefit, but such a deterministic choice is not affordable.

From the perspective of type 1, its purchased assignment is a lottery and its assignment into one or the other of the two active segregated exchanges is random. From the perspective of the economy-wide equilibrium a fixed fraction of type 1 are assigned to one or the other of the two active segregated exchanges, with fractions adding to one, and fractions equal to the probabilities from the perspective of the individual agent. We shall need a broker-dealer as an intermediary to allow this pooling to happen, and then we can define formally the necessary extension of the definition of competitive equilibrium with rights to trade in segregated exchanges.

Finally, note that the example is of course generating consumption allocations $\mathbf{c}$, saving k, and transfer $\boldsymbol{\tau}$ as excess demands. We could have mentioned security trades $\boldsymbol{\theta}$ as in Example 2, but here as in Example 1 there are no active security trades, so these were all zero. The discrepancy Δ in state s is simply a scaled version of excess demand in state s, as was made clear earlier. Likewise, the example is in terms of fundamental z but we could as easily rewritten the problem in terms of spot price $p(z)$. We continue to use Δ and z in the remainder of this section, but the reader should note yet again in the collateral environment there is an entirely equivalent formulation in terms of p; the only difference is notation.

Mixture Representation of the New Markets

To introduce a key piece of notation, recalling of course that we have a continuum of agents of each type, let $x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ be the fraction of agents type h assigned to a bundle $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$. At the individual level, for each agent type h, let $x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) \geq 0$ denote a probability measure on $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$. In other words, $x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ is the probability of receiving period $t = 0$ consumption $\mathbf{c}_0 \equiv (c_{10}, c_{20})$, collateral k, securities $\boldsymbol{\theta} \equiv [\theta_{\ell s}]_{\ell,s}$, spot trade $\boldsymbol{\tau} \equiv [\tau_{\ell s}]_{\ell,s}$, and being in exchanges indexed by $\mathbf{z} \equiv [z_s]_s$ with rights to trade $\boldsymbol{\Delta} \equiv [\Delta_s(z_s)]_s$.

All securities contracts are entered into ex-ante at $t = 0$, and spot trades and the valuation of collateral take place at spot price $p(z_s)$. Unlike the discrete choice δ^h, x^h notation in Section 3.2, it is not necessary to index all the objects in the commodity vector by $\mathbf{z}$. This is because of simple laws of probability: a probability of objects conditioned on $\mathbf{z}$ times the marginal probability of $\mathbf{z}$ can be rewritten as a joint probability of $\mathbf{z}$ and those objects. But this still allows many of the objects chosen to be degenerate, as in the example of Environment 4 presented above, where only agent type 1 chooses a lottery on $\mathbf{z}$. More generally, most of the objects in x^h here can be degenerate condition on $\mathbf{z}$. Securities conditioned on that draw of $\mathbf{z}$ are degenerate but again tied to $\mathbf{z}$, and if $\mathbf{z}$ is random as it is for type 1, then the security allocation is correspondingly random.

Each bundle $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ will be feasible for an agent type h only if the underlying collateral and security assignments satisfy the collateral constraints

$$p_s R_s k + \theta_{1s} + p_s \theta_{2s} \quad \geq \quad 0, \ \forall s, \tag{17}$$

the relationship among $\mathbf{z}$ and the $\Delta_s(z_s)$, $s = 1, 2, \ldots, S$ for each z_s:

$$\Delta_s(z_s) = \left(e^h_{2s} + R_s k\right)\left(z_s - \frac{e^h_{1s}}{e^h_{2s} + R_s k}\right), \tag{18}$$

and the spot trade $\boldsymbol{\tau}$ satisfies the spot budget constraint for each state s:

$$\tau_{1s} + p(z_s)\,\tau_{2s} = 0. \tag{19}$$

Accordingly, we impose the following condition on a probability measure:

$$\begin{aligned} x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) &\geq 0 \text{ if } (\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) \text{ satisfies (17)-(19)}, \\ &= 0 \text{ if otherwise.} \end{aligned} \tag{20}$$

More formally, the consumption possibility set of an agent type h is defined by

$$X^h = \left\{ \mathbf{x}^h \in \mathbb{R}^n_+ : \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}} x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) = 1, \text{ and (20) holds} \right\}. \tag{21}$$

Note that X^h is compact and convex. In addition, the non-emptiness of X^h is guaranteed by assigning mass one to each agent's endowment, i.e., no trade is a feasible option.

With all choice objects gridded up as an approximation, the commodity space L is assumed to be a finite n-dimensional linear space. We prove the existence and welfare theorems for a given, gridded space. But this is not at all essential. The limiting arguments under the weak-topology used in Prescott and Townsend (1984a) can be applied to establish existence and the welfare theorems for lotteries or measures on continuum spaces; even though the commodity space L is not finite, it is a separable metric space (Parthasarathy, 1967).

To decentralize, we need a valuation or pricing function in the same commodity space. So, let $P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ be the price for a commodity point $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$. However we can already guess that, apart from a normalization to express in terms of the numeraire good 1, in the equilibrium

$$P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) = c_{10} + p_0 c_{20} + p_0 k + \sum_{\ell, s} Q_\ell(z_s, s)\,\theta_{\ell s} + \sum_{\ell, s} p_\ell(z_s, s)\,\tau_{\ell s} + \sum_s P_\Delta(z_s, s)\,\Delta_s. \tag{22}$$

Thus we have a representation of prices on the objects separately; that is, on consumption goods at $t = 0$, security purchases or issues, spot market consumption, and the market rights. For convenience, though we retain the short-hand P notation.

For each consumer type h, the expected utility value derived from a bundle $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ is given by

$$V^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) = u(c_{10}, c_{20}) + \beta \sum_s \pi_s u\left(e^h_{1s} + \theta_{1s} + \tau_{1s}, e^h_{2s} + R_s k + \theta_{2s} + \tau_{2s}\right).$$

Thus, each agent type h chooses $\mathbf{x}^h$ in period $t = 0$ to maximize its expected utility, over states and the chosen lotteries,

$$\max_{\mathbf{x}^h} \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}} x^h \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right) V^h \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right) \tag{23}$$

subject to $\mathbf{x}^h \in X^h$, and period $t = 0$ budget constraint, that the valuation of endowments sold provides revenue for purchase of the lotteries.

$$\sum_{\mathbf{c}_0, k, \theta, \tau, \mathbf{z}, \boldsymbol{\Delta}} P \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right) x^h \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right) \quad \leq \quad e_{10}^h + p_0 e_{20}^h, \tag{24}$$

taking price of good 2 at $t = 0$, p_0, and prices of lottery, $P \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right)$ as given.

Here we now create the broker dealers as intermediaries producing trades and assignment to the exchanges. Formally, the broker-dealer issues (sells) $b \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right) \in \mathbb{R}_+$ units of each bundle $\left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right)$, at the unit price $P \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right)$. Note that $b \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right)$ at a particular bundle $\left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right)$ is the number of units of that bundle. There is nothing random. $b \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right)$ simply measures the quantity of the particular bundle sold. Another distinct bundle $\left(\mathbf{c}_0, k, \boldsymbol{\theta}', \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right)$ has its own quantity, number of units $b \left(\mathbf{c}_0, k, \boldsymbol{\theta}', \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right)$. With $\boldsymbol{\theta} \neq \boldsymbol{\theta}'$, the intermediary is taking distinct positions in the market. The clearing constraints below will ensure that when we add up over all bundles, the net positions add up to zero.

Let $\mathbf{b} \in L$ be the vector of the number of bundles issued as one move across the underlying commodity points $\left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right)$. With constant returns to scale (see below), the profit of a broker-dealer must be zero and the number of broker-dealers becomes indeterminate. Therefore, without loss of generality, we act as if there were one representative broker-dealer, which takes prices as given.

The objective of the broker-dealer is to maximize its profit by supplying $b \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right)$ as follows:

$$\max_{\mathbf{b}} \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}} b \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right) \left[P \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right) - c_{10} - p_0 c_{20} - p_0 k\right] \tag{25}$$

subject to clearing constraints:

$$\sum_{\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z}_{-s},\boldsymbol{\Delta}} b\left(\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z}_{-s},z_s,\boldsymbol{\Delta}\right)\theta_{\ell s}=0,\ \forall s;z_s;\ell=1,2, \tag{26}$$

$$\sum_{\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z}_{-s},\boldsymbol{\Delta}} b\left(\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z}_{-s},z_s,\boldsymbol{\Delta}\right)\tau_{\ell s}=0,\ \forall s;z_s;\ell=1,2, \tag{27}$$

$$\sum_{\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z}_{-s},\boldsymbol{\Delta}} b\left(\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z}_{-s},z_s,\boldsymbol{\Delta}\right)\Delta_s=0\ ,\forall s;z_s, \tag{28}$$

again taking prices $p_0, P\left(\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z},\boldsymbol{\Delta}\right)$ as given. Constraints such as (26) ensure that the books of the broker dealer be balanced, that the net position is zero. That is, the broker dealer is a clearing house as a buyer for every seller and a seller for every buyer. The broker dealer is in the middle of all trades in securities, spot markets, and rights to trade as in (26)-(28). Also, it must purchase enough of the consumption good $\mathbf{c}_0$ to be able to pay out consumption good and assign collateral k to promises, according to the quantities $b\left(\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z},\boldsymbol{\Delta}\right)$.

The existence of a maximum to the broker-dealer's problem requires, that for any bundle $\left(\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z},\boldsymbol{\Delta}\right)$,

$$P\left(\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z},\boldsymbol{\Delta}\right)\leq c_{10}+p_0c_{20}+p_0k+\sum_{\ell,s}\widehat{Q}_\ell\left(z_s,s\right)\theta_{\ell s}+\sum_{\ell,s}\widehat{p}_\ell\left(z_s,s\right)\tau_{\ell s}+\sum_{s}\widehat{P}_\Delta\left(z_s,s\right)\Delta_s, \tag{29}$$

where $\widehat{Q}_\ell\left(z_s,s\right)$, $\widehat{p}_\ell\left(z_s,s\right)$ and $\widehat{P}_\Delta(z_s,s)$ are the shadow price of an ex-ante security paying in good ℓ of constraints (26), the shadow price of the spot trade of good ℓ of constraints (27), and the shadow price of the right to trade in the security exchange z_s of constraints (28), respectively. This is where the prices of active and inactive exchanges come from. Condition (29) holds with equality if $b\left(\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z},\boldsymbol{\Delta}\right)>0$. In this case the revenue per unit trade in the $b\left(\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z},\boldsymbol{\Delta}\right)$ is equal to the sum of these shadow costs times the quantity of commitments in the particular objects. On the other hand, if the inequality (29) is strict, then $b\left(\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z},\boldsymbol{\Delta}\right)=0$ as when implicit shadow costs are greater than revenue. We can still price the inactive exchanges by setting (29) at equality.

Definition 2. A competitive equilibrium with segregated exchanges (with mixtures) is a specification of allocation $\left(\mathbf{x}^h,\mathbf{b}\right)$, and prices $\left(p_0, P\left(\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z},\boldsymbol{\Delta}\right)\right)$ such that

(i) for each h, $\mathbf{x}^h\in X^h$ solves utility maximization problem (23) subject to period $t=0$ budget constraint (24), taking prices as given;

(ii) for the broker-dealer, $\left\{\mathbf{b}, \widehat{Q}_\ell(z_s, s), \widehat{p}_\ell(z_s, s), \widehat{P}_\Delta(z_s, s)\right\}$ solve profit maximization problem (25) subject to clearing-trade constraints (26)-(28) taking prices as given;

(iii) market clearing condition for good 1 in period $t = 0$ holds:

$$\sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}} b(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})\, c_{10} = \sum_h \alpha^h e_{10}^h, \tag{30}$$

market clearing condition for good 2 in period $t = 0$ holds:

$$\sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}} b(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})\, [c_{20} + k] = \sum_h \alpha^h e_{20}^h, \tag{31}$$

and market clearing conditions for mixtures in period $t = 0$ hold:

$$\sum_h \alpha^h x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) = b(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}), \ \forall (\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}). \tag{32}$$

4 Mapping Economies into A Generalized Framework

The key point is that our market-based solution concept is applicable to many economies in which agents face a friction that generates constraints containing spot market prices. There are at least 6 prototype economies that fit into our framework. These include exogenous incomplete markets as in Section 2.1, collateral economy as in Section 2.2, fire sales economy (Lorenzoni, 2008), liquidity constrained economy (Hart and Zingales, 2013), moral hazard with retrading (Acemoglu and Simsek, 2012; Kilenthong and Townsend, 2011), and hidden information with retrading (Diamond and Dybvig, 1983).

The collateral economy and the incomplete markets economy have already been described in detailed in earlier sections of the paper. For each type h there are as many collateral or spot budget constraints as there are states S.

For the fire sales economy of Lorenzoni (2008), one key constraint that causes an inefficiency is a no default condition. This is similar to the one of the collateral economy, honoring claims to transfer, from negative positions. There are also spot market budget constraints, and all these can vary depending on who is a producer or consumer. There are 11 sets of obstacle-to-trade constraints.

The next economy is the liquidity constrained economy of Hart and Zingales (2013). There are two types of agents, doctors and builders. A storage claim on the numeraire good is the liquidity of the model. So there is no insurance and both types save to get their desired liquidity. This model features over saving.

For the moral hazard with retrading economy of Acemoglu and Simsek (2012) and Kilenthong and Townsend (2011), there are two goods produced by unobserved effort. The key constraints that cause inefficiency are incentive constraints coupled with retrading. The incentive constraints ensure that the agent takes the recommended action a and so $a' = a$. For each recommended action one has to guard against a' so there are $A \times A$ constraints, when A is the number of possible actions. High powered incentives are mitigated by trade in spot markets, and the incentive constraint is written in terms of the value function at spot market prices. Note that here there is ex-post diversity in realized types for a given ex ante type, even for a given action, as outputs can differ.

The last prototype economy is the hidden information economy of Diamond and Dybvig (1983). The truth-telling constraints ensure that the agent reports the true preference shock. If there are spot market trades in bonds in the interim period, truth-telling constraints are written in terms of a value function at spot prices. Here also there are ex post types, who is patient, or urgent.

Each of the example economies outlined above maps as a special case into the general economy. To keep the notation under control, we drop all the notations needed for private information economies in this paper. More details are available in Kilenthong and Townsend (2014a).

4.1 General Economy Environment: Commodity Space, Preferences, Endowments, and Technology

There are L commodities. These can be basic underlying commodities and also date and/or state contingent commodities where the date and/or state are public. There is a continuum of agents of measure one. The agents are divided into H (ex-ante) types, each of which is indexed by $h = 1, 2, \cdots, H$. Each type h consists of $\alpha^h \in [0, 1]$ fraction of the population such that $\sum_h \alpha^h = 1$. Each agent type h has an endowment $\mathbf{e}^h \in \mathbb{R}_+^L$. Note that $\mathbf{c}^h$ and $\mathbf{e}^h$ lie in the L-dimensional commodity space. The preferences of an agent of type h are represented by the utility function $U^h\left(\mathbf{c}^h\right)$, where $\mathbf{c}^h \in \mathbb{R}_+^L$ is the consumption allocation for an agent of type h.

Each agent of type h has access to a production technology defined implicitly by

$$\mathbf{F}^h\left(\mathbf{y}^h\right) \geq \mathbf{0}, \tag{33}$$

where $\mathbf{y}^h \in \mathbb{R}^L$ is the vector of its inputs and outputs in commodity space L. This production technology is generally a multidimensional vector of constraints. For example, the production technology in each state s for the collateral economy with inputs into storage y_{20}^h, negative, and outputs y_{2s}^h, positive, is

$$F_s\left(\mathbf{y}^h\right) = -y_{2s}^h - R_s y_{20}^h \geq 0, \forall s = 1, \ldots, S. \tag{34}$$

4.2 Market Structure: Security and Spot Markets

There are J securities. Let $\theta_j^h \in \mathbb{R}$ denote the amount of security j acquired (negative if sold) by an agent of type h, and $\mathbf{D}_j = [D_{j\ell}]_{\ell=1}^L \in \mathbb{R}_+^L$ denote its payoff vector. Thus securities have payoffs of goods in the L-dimensional space of underlying commodities. Notationally, let $\mathbf{D} = [\mathbf{D}_j]_{j=1}^J$ be the payoff matrix of all securities. Let $\mathbf{Q} \in \mathbb{R}_+^J$ be the price vector of all securities, that is, $Q_j \geq 0$ for $j = 1, \ldots, J$.

In addition, agents can trade in each of M spot markets of subsets of commodities. Let $L^m \subset L$ be the subset of commodities that can be traded in spot markets m. Let $\boldsymbol{\tau}^h$ denote the set of trades in these markets with $\tau_{\ell m}^h$ denoting the amount of good ℓ in market L^m acquired (negative if surrendered) by an agent of type h. Note again that these spot trades $\tau_{\ell m}^h$ are restricted to be traded with commodities in L^m only. Let $\mathbf{p}_m \equiv [p_{\ell m}]_{\ell \in L^m} \in \mathbb{R}_+^{L^m}$ be the price vector of commodities in L^m. For example, in the leading example economies, there are $M = S + 1$ sets of the spot markets, one in the first period and S of them in the second period.

The relationship between consumption, endowments, securities, spot trades, and outputs for an agent of type h is defined implicitly by

$$\mathbf{g}^h\left(\mathbf{c}^h, \mathbf{e}^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h, \mathbf{y}^h\right) = \mathbf{0}. \tag{35}$$

These are simply obvious identities or accounting formulas defining consumption as determined by the other objects. This condition again is generally a multidimensional vector of conditions, each of which is indexed by n. For example, the relationship for good 1 in each state $n = s$ for the collateral economy is

$$g_n^h \left(\mathbf{c}^h, \mathbf{e}^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right) = c_{1s}^h - e_{1s}^h - \theta_{1s}^h - \tau_{1s}^h = 0. \tag{36}$$

Similarly, the relationship for good 1 in state $n = s$ for the incomplete markets economy is

$$g_n^h \left(\mathbf{c}^h, \mathbf{e}^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right) = c_{1s}^h - e_{1s}^h - \sum_j D_{js}\theta_j^h - \tau_{1s}^h = 0, \tag{37}$$

where D_{js} is the payoff of asset j in units of good 1 in state s.

4.3 Trade Frictions: Obstacle-to-Trade Constraints

There are I sets of obstacle-to-trade constraints, each of which contains potentially multiple conditions, indexed by $a = 1, \ldots, A$. Each set of obstacle-to-trade constraints $i = 1, 2, ...I$ depends on the spot prices of a particular subset of commodities $\mathbf{p}^i$ or the prices of a particular subset of securities[12] denoted $\mathbf{Q}^i$, or both. Each can depend on the same set of prices $(\mathbf{p}^i, \mathbf{Q}^i)$. In their general form, each obstacle to trade constraint a in set i can be written as:

$$C_{i,a}^h \left(\mathbf{c}^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h, \mathbf{y}^h, \mathbf{p}^i, \mathbf{Q}^i\right) \geq 0, \text{ for } a = 1, \ldots, A; i = 1, \ldots, I; h = 1, \ldots, H. \tag{38}$$

Online Appendix I writes out these constraints for each of the six prototypes.

We also note that it is best to keep track of the constraints within a particular set i and the sets of constraints I separately. This notation also allows us do make clear the scalar in the collateral economy and the vector in the incomplete market economy. For example, in

[12]The dependency on market-clearing prices of these obstacle-to-trade constraints is the source of price externalities. Most of the literature, and the first two environments of this paper, focuses only on the dependency on the restricted/spot prices. This section explicitly puts security prices into the constraints in order to emphasize that price externalities could arise even when we shut down the spot markets. As shown in the collateral economy in online Appendix I, one can get rid of the spot markets there since they are redundant. The collateral constraints (the need to back promises by collateral) then depend on security prices only, but the price externality still occurs.

the collateral economy, there is only one condition $a = 1$ within each of a set of obstacle-to-trade constraints $s = 1, 2, \ldots, S$. For each one $i = s$, we write the price associated with the externality as $\mathbf{p}^i = p_s$. Namely,

$$C^h_{s,1}\left(\boldsymbol{\theta}^h, \mathbf{y}^h, p_s\right) = p_s y^h_{2s} + \theta^h_{1s} + p_s \theta^h_{2s} \geq 0, \forall h; s = 1, \ldots, S. \tag{39}$$

The state contingent p_s is an object of choice in exchanges and rights. The quantity of rights for the collateral economy is defined by $\Delta_s(p_s)$, again for each state s. Then $s = 1, 2, \ldots, S$ defines the set of constraints, one for each state s. This is an implication of the completeness of the markets, which allows us to separate each set of spot markets from each other as in (39).[13]

On the other hand, in the incomplete markets economy, there is a unique, single obstacle-to-trade constraint, but it is defined by a set of elements within it. There are $A = S$ conditions within the $i = 1$ set. For the set, we write $\mathbf{p}^i = \mathbf{p}$ with elements in the vector denoted $a = s$:

$$C^h_{1,s}\left(\boldsymbol{\tau}^h, \mathbf{p}\right) = \tau^h_{1s} + p_s \tau^h_{2s} = 0, \forall h; s = 1, \ldots, S. \tag{40}$$

Here vector $\mathbf{p}$ is an object of choice in exchanges and rights. The quantity of rights for the incomplete markets economy is denoted $\Delta_s(\mathbf{p})$, with arguments running over the elements s, and $\mathbf{p}$ is the common vector. Here in (40) a given security trade has implications for all of the S spot markets, and an agent is buying rights in each of them having specified the vector $\mathbf{p}$.

4.4 Welfare Theorems and Existence Theorem

As in the classical general equilibrium model, the economy is a well-defined convex economy, i.e., the commodity space is Euclidean, the consumption set is compact and convex, and the utility function is linear. As a result, the first and second welfare theorems hold, and a competitive equilibrium exists. In particular, this section proves that the competitive equilibrium is constrained optimal and any constrained optimal allocation can be supported

[13] See Kilenthong and Townsend (2011) for the obstacle-to-trade constraints in private information economies, which conceptually can be mapped into our general framework.

by a competitive equilibrium with transfers. Then, we will use Negishi's method to prove the existence of a competitive equilibrium.

The standard contradiction argument will be used to prove the following first welfare theorem. We also assume that there is local non-satiation point in the consumption set. Based on this local non-satiation assumption, we can prove the following:

Theorem 1. *With local non-satiation of preferences, a competitive equilibrium with segregated exchanges is constrained Pareto optimal.*

Proof. See Appendix B. □

The Second Welfare theorem states that any Pareto optimal allocation, corresponding to strictly positive Pareto weights, can be supported as a competitive equilibrium with transfers. The standard approach applies here. In particular, we first prove that any constrained optimal allocation can be decentralized as a compensated equilibrium. Then, we use a standard cheaper-point argument (see Debreu, 1954) to show that any compensated equilibrium is a competitive equilibrium with transfers.

Theorem 2. *Any Pareto optimal allocation corresponding with strictly positive Pareto weights $\lambda^h > 0, \forall h$ can be supported as a competitive equilibrium with segregated exchanges with transfers.*

Proof. See Appendix C. □

For existence of competitive equilibrium, we use Negishi's mapping method (Negishi, 1960). The proof benefits from the second welfare theorem. Specifically, a part of the mapping applies the theorem in that the solution to the Pareto program is a competitive equilibrium with transfers. We then show that a fixed-point of the mapping exists and represents a competitive equilibrium without transfers.

Theorem 3. *With local non-satiation of preferences and positive endowments, a competitive equilibrium with segregated exchanges exists.*

Proof. See Appendix D. □

In addition, we can show that in a classical economy without pecuniary externalities, the set of competitive equilibrium allocations does not change when segregated markets are introduced. More specifically, start in the extended commodity space with markets for rights at various prices, writing down the programming problem. Then guess that a solution to the first order conditions of the Lagrangian problem (which are both necessary and sufficient) is the solution, quantities and Lagrange multipliers, of the standard classical economy, without externalities, in the standard commodity space. Rights to trade in the extended commodity space are simply again the excess demands of the classical economy, and there are no additional obstacle to trade constraints in either. The guess is verified to be correct. This implies that the (spot) prices in an active segregated exchange must be the same as the shadow prices from the planning problem without the segregated exchanges. That, in turn, ensures that the Lagrange multipliers for the rights constraints for those active spot market exchanges have to be zero. In the analogue decentralized equilibrium, this implies that the prices of the rights to trade in those are zero as well. This makes sense since there is no reason to restrict ("tax" or "subsidize") trade, as that trade is not imposing an externality. However, the Lagrange multipliers or decentralized prices of the rights in inactive exchanges are not necessarily zero. In fact, they should not be zero to help guide agents to choose the optimal exchanges in equilibrium. This is what is preventing the emergence of new equilibria that might feature some kind of price discrimination.

5 Concluding Remarks

Here we draw on the insights of Coase (1960) and Lindahl (1958), extend the commodity space as in Arrow (1969), overcome some conceptual and technical hurdles, and show how the appropriate set of markets can eliminate fire sale externalities and the inefficiency of incomplete security markets. Our solution concept extends to many other well known environments in the literature. By its nature, a pecuniary externality has to do with the impact of prices in constraints beyond the role of prices in budget constraints, as happens in many models. The solution can be put rather simply: create segregated market exchanges which specify prices in advance (but with the same prices that also clear active markets ex post) and price the rights to trade in these markets so that participant types pay, or are compensated, consistent with the market exchange they choose and that type's excess demand contribution to the price in that exchange.

We conclude this paper where we began and come back to markets that seemingly have suffered from fire sales, such borrowing contracts backed by collateral. We do so in such a way as to review the lessons learned from this particular application and hence from the paper more generally.

In the complete markets version, with Arrow-Debreu securities, any security which defaults can be replaced by an equivalent one that does not, so default per se is not the issue. In our extension to remove externalities, we simply specify that, in addition to taking a position on the security/promise, buying or selling discount bonds in the securities market, ex ante, each trader also buys the rights ex ante to buy or sell the security in the subsequent spot market at particular specified prices. These transactions are related, but they are not the same thing. Our examples suggest that if an agent is a borrower in the securities market, in the initial market, with a promise to repay, a promised backed by the collateral good, then that agent has to buy in addition in the initial market the rights to sell that collateral good in the subsequent spot market. This is in addition to entering into the original borrowing contract.

Now, with incomplete markets, an agent is purchasing an entire vector of rights over the vector of possible spot market prices. Our example for the incomplete markets economy makes clear the fundamental economics is quite similar: in addition to ex ante security trades, an agent has to buy rights to sell the good in storage ex post in the states that motivated her use of that saving, to help cover adverse endowment realizations, though she may be selling rights in other states. Our main point is that in an extension to remove externalities for a hybrid collateral-incomplete-markets economy, we would still need the additional ex ante rights to unwind in spot markets over and above the original ex ante security sales.

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A Market-Based Solution for The General Model

This appendix formulates a market-based solution for the general model.

Let $\mathbf{\Delta}_{i,a}\left(\mathbf{p}^i, \mathbf{Q}^i\right)$ denote the vector of rights to trade in a particular security exchange with specified prices $\left(\mathbf{p}^i, \mathbf{Q}^i\right)$ associated with obstacle-to-trade constraint (i, a). This object is the externality-correcting commodity vector that, as in standard Walrasian equilibrium, will have its own unit of account market prices, $\mathbf{P}_{\Delta}\left(\mathbf{p}^i, \mathbf{Q}^i, i, a\right)$, $i = 1, 2, \ldots, I$ and $a = 1, \ldots, A$. As there can be multiple commodities, say L^a for each a, the rights associated with the constraint generally must be a vector. In order to be eligible to trade in an exchange $\left(\mathbf{p}^i, \mathbf{Q}^i\right)$, an agent of type h must hold the rights to trade $\mathbf{\Delta}_{i,a}\left(\mathbf{p}^i, \mathbf{Q}^i\right) = \mathbf{d}^h_{i,a}\left(\mathbf{e}^h, \theta^h, \mathbf{y}^h, \mathbf{p}^i, \mathbf{Q}^i\right)$, namely her excess demands for commodities associated with the constraint for all $a = 1, 2, \ldots, A$. For example, with all L^a commodities available to trade in the same spot markets, the rights associated to this constraint will be her excess demands for $L^a - 1$ commodities. Of course, each excess demand depends on agent's type, agent's endowment, agent's choices, and these prices. As there can be multiple sets of constraints, notationally, let $(\mathbf{p}, \mathbf{Q}) \equiv \left[\mathbf{p}^i, \mathbf{Q}^i\right]_i$ and $\mathbf{\Delta}$ be the associated vector of the rights.

As a generalized version of (20), let $x^h\left(\mathbf{c}, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{y}, \mathbf{p}, \mathbf{Q}, \mathbf{\Delta}\right)$ be the probability measure on $\left(\mathbf{c}, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{y}, \mathbf{p}, \mathbf{Q}, \mathbf{\Delta}\right)$ for an agent of type h. For convenience in notation, we have deleted type h from the argument in these. In other words, $x^h\left(\mathbf{c}, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{y}, \mathbf{p}, \mathbf{Q}, \mathbf{\Delta}\right)$ is the probability of receiving allocation $\left(\mathbf{c}, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{y}\right)$, and being in exchanges $(\mathbf{p}, \mathbf{Q})$ with the rights to trade $\mathbf{\Delta}$. We now consider a bundle $\left(\mathbf{c}, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{y}, \mathbf{p}, \mathbf{Q}, \mathbf{\Delta}\right)$ as a typical underlying commodity bundle. For notational purposes, let $\mathbf{w} \equiv \left(\mathbf{c}, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{y}, \mathbf{p}, \mathbf{Q}, \mathbf{\Delta}\right)$ be a typical bundle. As in Section 3.4, let $P(\mathbf{w})$ be the price of a bundle or contract $\mathbf{w}$.

Each bundle will be feasible for an agent of type h with endowment $\mathbf{e}^h$ only if it satisfies the constraints as in Section 4.2, namely the spot-market budget constraints, the consumption-relationship constraints, the technology constraints, the obstacle-to-trade constraints, and the right-to-trade requirements[14]:

$$\sum_{\ell=1}^{L^m} p_{\ell m}\tau_{\ell m} \leq 0, \forall m, \tag{41}$$

$$\mathbf{g}^h\left(\mathbf{c}, \mathbf{e}^h, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{y}\right) = \mathbf{0}, \tag{42}$$

$$\mathbf{F}^h\left(\mathbf{y}\right) = \mathbf{0}, \tag{43}$$

$$C^h_{i,a}\left(\mathbf{c}, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{y}, \mathbf{p}^i, \mathbf{Q}^i\right) \geq 0, \forall i, a \tag{44}$$

$$\boldsymbol{\Delta}_{i,a}\left(\mathbf{p}^i, \mathbf{Q}^i\right) - \mathbf{d}^h_{i,a}\left(\mathbf{e}^h, \boldsymbol{\theta}^h, \mathbf{y}^h, \mathbf{p}^i, \mathbf{Q}^i\right) = \mathbf{0}, \forall i, a. \tag{45}$$

Let $\mathbf{x}^h \equiv \left[x^h\left(\mathbf{w}\right)\right]_{\mathbf{w}}$ be a typical lottery for an agent of type h. As a probability measure, a lottery of an agent of type h, x^h, satisfies the following probability constraint:

$$\sum_{\mathbf{w}} x^h\left(\mathbf{w}\right) = 1. \tag{46}$$

Accordingly, we impose the following condition on the probability measure as follows:

$$\begin{aligned} x^h\left(\mathbf{c}, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{y}, \mathbf{p}, \mathbf{Q}, \boldsymbol{\Delta}\right) &\geq 0 \text{ if conditions (41)-(45) hold,} \\ &= 0 \text{ if otherwise.} \end{aligned} \tag{47}$$

We denote the set of feasible vector $\mathbf{x}^h$ as $X^h = \left\{\mathbf{x}^h \in \mathbb{R}^n_+ : (46) \text{ and } (47) \text{ hold}\right\}$.

Consumers: Each agent of type h, taking prices $P\left(\mathbf{w}\right)$ as given, chooses $\mathbf{x}^h$ to maximize its expected utility:

$$\max_{\mathbf{x}^h \in X^h} \sum_{\mathbf{w}} x^h\left(\mathbf{w}\right) U^h\left(\mathbf{c}\right) \tag{48}$$

subject to the budget constraint

$$\sum_{\mathbf{w}} P\left(\mathbf{w}\right) x^h\left(\mathbf{w}\right) \leq 0. \tag{49}$$

Financial Intermediaries: As in Section 3.4, there is a representative financial intermediary who issues (sells) $b\left(\mathbf{c}, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{y}, \mathbf{p}, \mathbf{Q}, \boldsymbol{\Delta}\right) \equiv b\left(\mathbf{w}\right) \in \mathbb{R}_+$ units of each bundle at the unit price $P\left(\mathbf{w}\right)$. Let $\mathbf{b}$ be the vector of the number of bundles issued as one moves across bundles $\mathbf{w}$.

[14]In some cases, as in a moral hazard with retrading and a hidden information with retrading, the right-to-trade requirements may be embedded implicitly in the obstacle-to-trade constraints. As a result, they may not be explicitly written out as separate constraints, as in Kilenthong and Townsend (2011).

Let L_0 be the set of commodities that are not part of any obstacle-to-trade constraints. For example, in both collateral and incomplete markets economies, L_0 contains two commodities, good 1 and good 2 in period $t = 0$. Similarly, let J_0 be the set of securities that are not part of any obstacle-to-trade constraints. Let L_i and J_i be the set of commodities and securities, respectively, associated with a set of obstacle-to-trade constraints i.

The intermediary's profit maximization problem is as follows:

$$\max_{\mathbf{b}} \sum_{\mathbf{w}} P(\mathbf{w}) \, b(\mathbf{w}) \tag{50}$$

subject to the clearing constraints

$$\sum_{\mathbf{w}} b(\mathbf{w}) \, \theta_j \leq 0, \forall j \in J_0, \tag{51}$$

$$\sum_{(\mathbf{c},\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{y},\mathbf{p}_{-i},\mathbf{Q}_{-i},\boldsymbol{\Delta})} b(\mathbf{w}) \, \theta_j \leq 0, \forall j \in J_i, i, \mathbf{p}^i, \mathbf{Q}^i, \tag{52}$$

$$\sum_{\mathbf{w}} b(\mathbf{w}) \, \tau_{\ell m} = 0, \forall \ell m \in L_0, \tag{53}$$

$$\sum_{(\mathbf{c},\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{y},\mathbf{p}_{-i},\mathbf{Q}_{-i},\boldsymbol{\Delta})} b(\mathbf{w}) \, \tau_{\ell m} = 0, \forall \ell m \in L_i, i, \mathbf{p}^i, \mathbf{Q}^i, \tag{54}$$

$$\sum_{(\mathbf{c},\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{y},\mathbf{p}_{-i},\mathbf{Q}_{-i},\boldsymbol{\Delta})} b(\mathbf{w}) \, \boldsymbol{\Delta}_{i,a} \left(\mathbf{p}^i, \mathbf{Q}^i\right) = \mathbf{0}, \forall i, a, \mathbf{p}^i, \mathbf{Q}^i, \tag{55}$$

where $\mathbf{p} = (\mathbf{p}^i, \mathbf{p}_{-i})$, and $\mathbf{Q} = (\mathbf{Q}^i, \mathbf{Q}_{-i})$. These constraints state that the financial intermediary must put together deals that execute all securities, spot trades, and rights to trade properly.

Market Clearing: The market-clearing conditions for contracts/lotteries are as follows:

$$\sum_{h} \alpha^h x^h(\mathbf{w}) = b(\mathbf{w}), \ \forall \mathbf{w}. \tag{56}$$

Definition 3. A competitive equilibrium with segregated exchanges is a specification of allocation $(\mathbf{x}, \mathbf{b})$, and prices $P(\mathbf{w})$ such that

(i) for each agent of type h, $\mathbf{x}^h \in X^h$ solves (48) subject to (49), taking prices as given;

(ii) for the financial intermediary, $\mathbf{b}$ solves (50) subject to (51)-(55), taking prices as given;

(iii) markets for contracts/lotteries clear; that is, (56) holds.

Constrained Optimal Allocations

A constrained optimal allocation is an attainable allocation such that there is no other attainable allocation that can make at least one agent type strictly better off without making any other agent type worse off. We characterize constrained optimality using the following Pareto program. Let $\lambda^h \geq 0$ be the Pareto weight of agent type h. There is no loss of generality to normalize the weights such that $\sum_h \lambda^h = 1$. A constrained Pareto optimal allocation $\mathbf{x} \equiv \left(\mathbf{x}^h\right)_{h=1}^{H} \in X^1 \times \ldots \times X^H$ solves the following Pareto program.

Program 1.

$$\max_{\mathbf{x}} \sum_h \lambda^h \alpha^h \sum_{\mathbf{w}} x^h(\mathbf{w}) U^h(\mathbf{c}) \tag{57}$$

subject to

$$\sum_{\mathbf{w}} \sum_h \alpha^h x^h(\mathbf{w}) \theta_j \leq 0, \forall j \in J_0, \tag{58}$$

$$\sum_{(\mathbf{c},\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{y},\mathbf{p}_{-i},\mathbf{Q}_{-i},\boldsymbol{\Delta})} \sum_h \alpha^h x^h(\mathbf{w}) \theta_j \leq 0, \forall j \in J_i, i, \mathbf{p}^i, \mathbf{Q}^i, \tag{59}$$

$$\sum_{\mathbf{w}} \sum_h \alpha^h x^h(\mathbf{w}) \tau_{\ell m} = 0, \forall \ell m \in L_0, \tag{60}$$

$$\sum_{(\mathbf{c},\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{y},\mathbf{p}_{-i},\mathbf{Q}_{-i},\boldsymbol{\Delta})} \sum_h \alpha^h x^h(\mathbf{w}) \tau_{\ell m} = 0, \forall \ell m \in L_i, i, \mathbf{p}^i, \mathbf{Q}^i, \tag{61}$$

$$\sum_{(\mathbf{c},\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{y},\mathbf{p}_{-i},\mathbf{Q}_{-i},\boldsymbol{\Delta})} \sum_h \alpha^h x^h(\mathbf{w}) \boldsymbol{\Delta}_{i,a}\left(\mathbf{p}^i, \mathbf{Q}^i\right) = \mathbf{0}, \forall i, a, \mathbf{p}^i, \mathbf{Q}^i, \tag{62}$$

It is clear that the objective function now is linear in $\mathbf{x}$. Thus, it is continuous and weakly concave. The feasible set X is non-empty, compact, and convex. Therefore, a solution to the Pareto program for given positive Pareto weights exists and is a global maximum.

B Proof of The First Welfare Theorem

Proof of Theorem 1. This proof follows Prescott and Townsend (1984a). Let allocations $(\mathbf{x}, \mathbf{b})$, and prices $P(\mathbf{w})$ be a competitive equilibrium. Suppose the competitive equilibrium allocation is not Pareto optimal, i.e., there is an attainable allocation $\tilde{\mathbf{x}}$ such that $\sum_{\mathbf{w}} \tilde{x}^h(\mathbf{w}) U^h(\mathbf{c}) \geq \sum_{\mathbf{w}} x^h(\mathbf{w}) U^h(\mathbf{c})$ for all h and $\sum_{\mathbf{w}} \tilde{x}^{\hat{h}}(\mathbf{w}) U^{\hat{h}}(\mathbf{c}) > \sum_{\mathbf{w}} x^{\hat{h}}(\mathbf{w}) U^{\hat{h}}(\mathbf{c})$ for some $\hat{h}$. With local non-satiation of preferences, $\sum_{\mathbf{w}} P(\mathbf{w}) x^h(\mathbf{w}) \leq \sum_{\mathbf{w}} P(\mathbf{w}) \tilde{x}^h(\mathbf{w})$

for all h, and $\sum_{\mathbf{w}} P(\mathbf{w}) x^{\hat{h}}(\mathbf{w}) < \sum_{\mathbf{w}} P(\mathbf{w}) \tilde{x}^{\hat{h}}(\mathbf{w})$ for some $\hat{h}$. Summing over all agents with weights α^h, we have

$$\sum_{\mathbf{w}} P(\mathbf{w}) \sum_h \alpha^h x^h(\mathbf{w}) < \sum_{\mathbf{w}} P(\mathbf{w}) \sum_h \alpha^h \tilde{x}^h(\mathbf{w}). \tag{63}$$

In addition, for each allocation $\mathbf{x}$ and $\tilde{\mathbf{x}}$, we can find a corresponding supply from the intermediary such that $b(\mathbf{w}) = \sum_h \alpha^h x^h(\mathbf{w})$ and $\tilde{b}(\mathbf{w}) = \sum_h \alpha^h \tilde{x}^h(\mathbf{w})$. Since both $\mathbf{x}$ and $\tilde{\mathbf{x}}$ satisfy all feasibility conditions, $\mathbf{b}$ and $\tilde{\mathbf{b}}$ both satisfy the clearing constraints (58)-(62). As a result, (63) can be rewritten as

$$\sum_{\mathbf{w}} P(\mathbf{w}) b(\mathbf{w}) < \sum_{\mathbf{w}} P(\mathbf{w}) \tilde{b}(\mathbf{w}). \tag{64}$$

On the other hand, the intermediary's profit maximization implies that

$$\sum_{\mathbf{w}} P(\mathbf{w}) b(\mathbf{w}) \geq \sum_{\mathbf{w}} P(\mathbf{w}) \tilde{b}(\mathbf{w}). \tag{65}$$

This is a contradiction! □

C Proof of The Second Welfare Theorem

Proof of Theorem 2. We will first prove that any constrained optimal allocation can be decentralized as a compensated equilibrium. Then, we will use a standard cheaper-point argument (see Debreu, 1954) to show that any compensated equilibrium is a competitive equilibrium with transfers. The compensated equilibrium is defined as follows.

Definition 4. A compensated equilibrium is a specification of allocation $(\mathbf{x}, \mathbf{b})$, and prices $P(\mathbf{w})$ such that

(i) for each h as a price taker, $\mathbf{x}^h \in X^h$ solves

$$\min_{\hat{\mathbf{x}}^h \in X^h} \sum_{\mathbf{w}} P(\mathbf{w}) \hat{x}^h(\mathbf{w}) \tag{66}$$

subject to

$$\sum_{\mathbf{w}} \hat{x}^h(\mathbf{w}) U^h(\mathbf{c}) \geq \sum_{\mathbf{w}} x^h(\mathbf{w}) U^h(\mathbf{c}); \tag{67}$$

(ii) for the financial intermediary, $\mathbf{b}$ solves (50) subject to (51)-(55), taking prices as given;

(iii) markets for contracts/lotteries clear; that is, (56) holds.

Given that the optimization problems are well-defined concave problems, Kuhn-Tucker conditions are necessary and sufficient. The proof is divided into three steps.

(i) Kuhn-Tucker conditions for a compensated equilibrium allocation: Let $\hat{\gamma}_U^h$ and $\hat{\gamma}_l^h$ be the Lagrange multiplier for constraint (67), and for the probability constraint (46). The optimal condition for $x^h(\mathbf{w})$ is given by

$$\hat{\gamma}_U^h U^h(\mathbf{c}) \leq P(\mathbf{w}) + \hat{\gamma}_l^h, \tag{68}$$

where the inequality holds with equality if $x^h(\mathbf{w}) > 0$. The optimal condition for the intermediary's profit maximization problem implies that, for any typical bundle $\mathbf{w}$,

$$\begin{aligned} P(\mathbf{w}) \leq \sum_{j\in J_0} \widehat{Q}_{j0}\theta_j + \sum_i \sum_{j\in J_i} \sum_{(\mathbf{p}^i,\mathbf{Q}^i)} \widehat{Q}_{ij}\left(\mathbf{p}^i,\mathbf{Q}^i\right)\theta_j + \sum_i \sum_{\ell m\in L_i} \sum_{(\mathbf{p}^i,\mathbf{Q}^i)} \hat{p}_{i\ell m}\left(\mathbf{p}^i,\mathbf{Q}^i\right)\tau_{\ell m} \\ + \sum_{\ell m\in L_0} \hat{p}_{\ell m}\tau_{\ell m} + \sum_i \sum_a \sum_{(\mathbf{p}^i,\mathbf{Q}^i)} \widehat{\mathbf{P}}_\Delta\left(\mathbf{p}^i,\mathbf{Q}^i,i,a\right)\cdot \boldsymbol{\Delta}_{i,a}\left(\mathbf{p}^i,\mathbf{Q}^i\right), \end{aligned} \tag{69}$$

where $\widehat{Q}_{j0}$, $\widehat{Q}_{ij}(\mathbf{p}^i,\mathbf{Q}^i)$, $\hat{p}_{\ell m}$, $\hat{p}_{i\ell m}(\mathbf{p}^i,\mathbf{Q}^i)$ and $\widehat{\mathbf{P}}_\Delta(\mathbf{p}^i,\mathbf{Q}^i,i,a)$, are the Lagrange multipliers for constraints (51)-(55), respectively, and " $\cdot$ " represents the standard inner product. The condition holds with equality if $b(\mathbf{w}) > 0$.

(ii) Kuhn-Tucker conditions for Pareto optimal allocations: A solution to the Pareto program satisfies following condition

$$\begin{aligned} \lambda^h U^h(\mathbf{c}) \leq \sum_{j\in J_0} \widetilde{Q}_{j0}\theta_j + \sum_i \sum_{j\in J_i} \sum_{(\mathbf{p}^i,\mathbf{Q}^i)} \widetilde{Q}_{ij}\left(\mathbf{p}^i,\mathbf{Q}^i\right)\theta_j + \sum_i \sum_{\ell m\in L_i} \sum_{(\mathbf{p}^i,\mathbf{Q}^i)} \tilde{p}_{i\ell m}\left(\mathbf{p}^i,\mathbf{Q}^i\right)\tau_{\ell m} \\ + \sum_{\ell m\in L_0} \tilde{p}_{\ell m}\tau_{\ell m} + \sum_i \sum_a \sum_{(\mathbf{p}^i,\mathbf{Q}^i)} \widetilde{\mathbf{P}}_\Delta\left(\mathbf{p}^i,\mathbf{Q}^i,i,a\right)\cdot \boldsymbol{\Delta}_{i,a}\left(\mathbf{p}^i,\mathbf{Q}^i\right) + \tilde{\gamma}_l^h, \end{aligned} \tag{70}$$

where $\tilde{\gamma}_l^h$ is the Lagrange multiplier for the probability constraint (46), and $\widetilde{Q}_{j0}$, $\widetilde{Q}_{ij}(\mathbf{p}^i,\mathbf{Q}^i)$, $\tilde{p}_{\ell m}$, $\tilde{p}_{i\ell m}(\mathbf{p}^i,\mathbf{Q}^i)$ and $\widetilde{\mathbf{P}}_\Delta(\mathbf{p}^i,\mathbf{Q}^i,i,a)$, are the Lagrange multipliers for constraints (58)-(62), respectively.

(iii) Matching dual variables and prices: Without loss of generality, let $\tilde{p}_{10}$ be the Lagrange multiplier of the spot trade of the numeraire good, which is in L_0. For example, we set good 1 in $t=0$ as the numeraire in our leading example economies. We then set $\hat{\gamma}_U^h = \frac{\lambda^h}{\tilde{p}_{10}}$, $\widehat{Q}_{j0} = \frac{\widetilde{Q}_{j0}}{\tilde{p}_{10}}$, $\widehat{Q}_{ij}\left(\mathbf{p}^i, \mathbf{Q}^i\right) = \frac{\widetilde{Q}_{ij}\left(\mathbf{p}^i,\mathbf{Q}^i\right)}{\tilde{p}_{10}}$, $\hat{p}_{\ell m} = \frac{\tilde{p}_{\ell m}}{\tilde{p}_{10}}$, $\hat{p}_{i\ell m}\left(\mathbf{p}^i, \mathbf{Q}^i\right) = \frac{\tilde{p}_{i\ell m}\left(\mathbf{p}^i,\mathbf{Q}^i\right)}{\tilde{p}_{10}}$, $\widehat{\mathbf{P}}_\Delta\left(\mathbf{p}^i, \mathbf{Q}^i, i, a\right) = \frac{\widetilde{\mathbf{P}}_\Delta\left(\mathbf{p}^i,\mathbf{Q}^i,i,a\right)}{\tilde{p}_{10}}$, and $\hat{\gamma}_l^h = \frac{\tilde{\gamma}_l^h}{\tilde{p}_{10}}$. These matching conditions imply that the optimal conditions of the Pareto program are equivalent to the optimal conditions for consumers' and market-maker's problems in the compensated equilibrium. To sum up, any Pareto optimal allocation is a compensated equilibrium.

We can show that any compensated equilibrium, corresponding to $\lambda^h > 0$, is a competitive equilibrium with transfers using the cheaper point argument, which is obvious given the strictly positive Pareto weight and strictly positive endowment. Using the cheaper-point argument, a compensated equilibrium is a competitive equilibrium with transfers. $\square$

D Proof of The Existence Theorem

Proof of Theorem 3. For notational convenience, we put the endowment $\mathbf{e}^h$ onto the grid. Let $\mathbf{P} = \left[P\left(\mathbf{w}\right)\right]_{\mathbf{w}}$ be the prices of all bundles. As in Prescott and Townsend (2005), with the possibility of negative prices, we restrict prices $\mathbf{P}$ to the closed unit ball;

$$D = \left\{\mathbf{P} \in \mathbb{R}^n | \sqrt{\mathbf{P}\cdot\mathbf{P}} \leq 1\right\}, \tag{71}$$

where "$\cdot$" is the inner product operator. Note that the set D is compact and convex.

Consider the following mapping $(\lambda, \mathbf{x}, \mathbf{P}) \to (\lambda', \mathbf{x}', \mathbf{P}')$, where $\lambda, \lambda' \in S^{H-1}$, $\mathbf{x}^h \in X^h$. Recall that the consumption possibility set X^h is non-empty, convex, and compact. Let $\overline{X}$ be the cross-product over h of X^h: $\overline{X} = X^1 \times \ldots \times X^H$.

The first part of the mapping is given by $\lambda \longrightarrow (\mathbf{x}', \mathbf{P}')$, where $\mathbf{x}'$ is the solution to the Pareto program given the Pareto weight λ, and $\mathbf{P}'$ is the renormalized prices. With the second welfare theorem, the solution to the Pareto program for a given Pareto weight λ also gives us (compensated) equilibrium prices $\mathbf{P}^*$. The local non-satiation of preferences implies that $\mathbf{P}^* \neq 0$. The normalized prices are given by

$$\mathbf{P}' = \frac{\mathbf{P}^*}{\mathbf{P}^*\cdot\mathbf{P}^*}.$$

Note that $\mathbf{P}'\cdot\mathbf{P}' = 1$. In order to preserve the convexity of the mapping with prices in the unit ball D, we define the convex hull of the normalized prices. Let $\widetilde{D}$ be the sets of all normalized prices, and accordingly $co\widetilde{D}$ be its convex hull. Since $\mathbf{P}' \in \widetilde{D}$, $\mathbf{P}' \in co\widetilde{D}$, which is compact and convex. Note that extending $\widetilde{D}$ to its convex hull does not add any new relative prices. It is not too difficult to show that this mapping, $\lambda \longrightarrow (\mathbf{x}', \mathbf{P}')$, is non-empty, compact-valued, convex-valued. By the Maximum theorem, it is upper hemi-continuous. In addition, the upper hemi-continuity is preserved under the convex-hull operation.

The second part of the mapping is given by $(\lambda, \mathbf{x}, \mathbf{P}) \longrightarrow \lambda'$. The new weight can be formed as follows:

$$\hat{\lambda}^h = \max \left\{ 0, \lambda^h + \frac{\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right)}{A} \right\}, \tag{72}$$

$$\lambda'^h = \frac{\hat{\lambda}^h}{\sum_h \hat{\lambda}^h}, \tag{73}$$

where A is a positive number such that $\sum_h \left|\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right)\right| \leq A$. It is clear that this mapping is also non-empty, compact-valued, convex-valued, and upper hemi-continuous. In conclusion, $(\lambda, \mathbf{x}, \mathbf{P}) \to (\lambda', \mathbf{x}', \mathbf{P}')$ is a mapping from $S^{H-1} \times \overline{X} \times S^{n-1} \longrightarrow S^{H-1} \times \overline{X} \times S^{n+1}$. Since each set is non-empty, compact, and convex, so is its cross-product. In addition, the overall mapping is non-empty, compact-valued, convex-valued, and upper hemi-continuous since these properties are preserved under the cross product operation. By Kakutani's fixed point theorem, there exists a fixed point $(\lambda, \mathbf{x}, \mathbf{P})$.

Proved in Theorem 2, any Pareto optimal allocation can be supported as a compensated equilibrium. In addition, the nonsatiation and the positive endowment assumptions ensure that there is a cheaper point as in the proof of Theorem 2. As a result, a compensated equilibrium is a competitive equilibrium with transfers.

We now need to show that there is no need for wealth transfers in equilibrium, i.e., the budget constraint without transfers $\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) = 0$ holds for every agent h. It is not difficult to show that $\sum_h \alpha^h \mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) = 0$ In addition, at a fixed point $\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right)$ must be the same sign for every h. Hence, $\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) = 0$ for every agent h. This clearly confirms that the budget constraint (without transfers) of every agent h holds. Hence, a competitive equilibrium (without transfers) exists. □