

# Risk-taking over the Life Cycle: Aggregate and Distributive Implications of Entrepreneurial Risk

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## Abstract

We study the risk-taking behavior of entrepreneurs in an environment with two main ingredients: finite lives and uninsurable idiosyncratic risk on the business. We show that the fraction of wealth invested in the business depends on the idiosyncratic risk premium and that it declines substantially over the life cycle. The consumption-wealth ratio is U-shaped over the life cycle. We solve for the wealth distribution both across and within age groups. We show that the variance of wealth conditional on age has an inverted-U shape, initially increasing with age and eventually declining. We find support for these predictions in the data using a survey of entrepreneurial activity in Thailand. We also consider the impact of financial development and demographic transitions on asset prices, economic activity, and inequality. We show that an increase in the fraction of idiosyncratic risk entrepreneurs can insure or a decline in population growth will lead to a reduction in the idiosyncratic risk premium, an increase in the capital stock of the economy, and a decline in inequality.

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# 1 Introduction

Entrepreneurship is inherently a risky activity, even more so in the context of developing countries. The decision to become an entrepreneur as well as the choice of the appropriate scale to operate will depend on how risky a given activity is, the expected return on the business, and the capacity or willingness of the entrepreneur to bear such risks. Hence, in order to understand the behavior of entrepreneurs and the impact of entrepreneurship on the economy, it is important to understand the implications of entrepreneurial risk. In this paper, we propose a framework to analyze the aggregate and distributive implications of such risks.

We start by documenting a substantial variation in the entrepreneurs' exposure to their businesses. Using data on entrepreneurs for rural and semi-urban villages in Thailand, we show that the share of financial wealth invested in the business declines sharply over the life cycle, with young entrepreneurs having a share of wealth invested in the business up to 60% higher than old entrepreneurs. Moreover, the level of financial wealth varies significantly both within and across age groups. In order to capture such heterogeneity, we consider a life cycle model where entrepreneurs face a portfolio choice of how much to invest in the business and how much to invest in a safe low return alternative.

The heterogeneity in risk-taking behavior over the life cycle is by no means accidental. We show that the effective risk aversion of an entrepreneur, i.e. the curvature of her value function, will vary over the life cycle even if the instantaneous utility function is of the CRRA type. In particular, the effective risk aversion will be decreasing in the *human-financial-wealth ratio* of an entrepreneur. Differences in human wealth, the present discounted value of non-business income, will then induce differences in risk-taking.<sup>1</sup>

In order to test whether this mechanism can generate the amount of heterogeneity we observe in the data, we construct an empirical measure of human wealth. An important aspect of this calculation is the choice of the discount rate. Given that labor income is risky, both in the data and in the model, the risk free interest rate is not the appropriate discount rate. We show how to use data on returns on a portfolio of business activities to find the correct risk-adjusted discount rate.<sup>2</sup> Given our measure of human wealth, we show that the human-financial wealth ratio declines over the life cycle. Moreover, the resulting increase in effective risk aversion is enough to quantitatively explain the decline in the exposure to the business over the life cycle.

The human-financial wealth ratio, through its impact on the effective risk aversion, determines how an entrepreneur evaluates a risk-return trade-off. Given the effective risk aversion, the decision of how much to be exposed to the business will depend on the actual expected return and volatilities. [Samphantharak and Townsend \(2018\)](#) documents that returns have an important *idiosyncratic* component. More than 90% of the variance is accounted for idiosyncratic risk. Interestingly, idiosyncratic risk accounts for only half of the expected returns, so the Sharpe ratio of aggregate risk is three times the one for idiosyncratic risk. We introduce these features into the model by assuming that returns are subject to aggregate and idiosyncratic risk. Given the differences in volatilities, we are able to generate

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<sup>1</sup>This result is reminiscent of the work on portfolio choice with labor income of [Bodie et al. \(1992\)](#) and [Viceira \(2001\)](#). See also [Heaton and Lucas \(1997\)](#) and [Koo \(1998\)](#)

<sup>2</sup>See [Huggett and Kaplan \(2016\)](#) for a similar approach on valuing human wealth.

endogenously the differences in returns and Sharpe ratios.

Introducing idiosyncratic risk into the model opens the door to a potential moral hazard problem, as idiosyncratic shocks are typically hard to monitor. We assume that aggregate shocks are public information, but idiosyncratic shocks are private information to the entrepreneur. This will limit the amount of idiosyncratic insurance an entrepreneur can contract. Entrepreneurs will be subject to a *skin-in-the-game constraint* and will be able to insure at most a fraction  $0 \leq \phi \leq 1$  of idiosyncratic risk, where  $\phi = 1$  amounts to full insurance and  $\phi = 0$  financial autarky. In contrast, entrepreneurs are allowed to buy aggregate insurance freely.

The Lagrange multiplier on this skin-in-the-game constraint plays an important role in the entrepreneurs' risk-taking decision. It turns out that the risk-taking decision of the entrepreneur depends only on the effective risk aversion, the level of idiosyncratic risk, and this multiplier, which we refer to as the *shadow price of idiosyncratic insurance*. In order to test the theory, it is then crucial to come up with a measure of such shadow price. We show that, given the skin-in-the-game parameter  $\phi$ , one can identify the shadow price of idiosyncratic insurance from the intercept of the time-series regression of returns of a given entrepreneur on the return of a portfolio of all businesses of a given region. The slope of this regression gives information about the aggregate risk premium. This is analogous to the  $\alpha$  and  $\beta$  of a CAPM regression of returns on the market portfolio.<sup>3</sup>

The level of aggregate and idiosyncratic returns, together with the human-financial wealth ratio, are also important to understand entrepreneur's savings behavior. We document that the consumption-wealth ratio has a U-shaped pattern over the life cycle, initially decreasing and eventually increasing by the end of the cycle. The model is able to replicate this pattern by a combination of two forces. First, the decline in the human-financial wealth ratio tend to reduce the rate of consumption. On the other hand, the marginal propensity to consume (MPC) increases with age, which tends to increase the consumption-wealth ratio. The first force dominates early in life, while the second dominates as the end of life approaches. The level of consumption-wealth ratio will depend on preferences and the return on the portfolio.

Having determined the risk-taking and consumption decisions of entrepreneurs, we can solve for the whole distribution of wealth in the economy. In particular, we can analyze how the presence of uninsurable idiosyncratic risk can affect wealth inequality and eventually asset prices and the level of economic activity. First, we consider the determination of between-age-group inequality, i.e. differences on average wealth of entrepreneurs of different ages. Average wealth has an inverted-U shape over the life cycle, as it initially increases with age and eventually falls at older ages. This is a result of having more time to accumulate wealth, and the higher the level of aggregate and idiosyncratic risk premium, the faster the entrepreneur accumulates, combined with an increase in the MPC with age, especially by the end of the life cycle. We test these predictions in the data and find similar life cycle pattern and overall level of between-group inequality.

We also consider the within-group inequality, i.e., differences in the level of wealth for entrepreneurs of a given age. The evolution of the distribution with age follows a partial differential equation, the so-called *Kolmogorov Forward Equation*. We are able to solve for the distribution of wealth conditional

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<sup>3</sup>Samphantharak and Townsend (2018) studied extensively these regression and characterized the empirical behavior of  $\alpha$ , but without giving an explicit structural interpretation to the regression.

on age in closed-form for the case where the entrepreneur leave no bequests. In particular, we are able to show that the variance of wealth also has an inverted U-shape pattern, it initially increases with age and then eventually decreases. We document that the variance of wealth follows a similar life cycle pattern in the Thai data.

Finally, we discuss the determination of equilibrium prices and consider a few counterfactuals exercises. First, we consider the impact of changes in the skin-in-the-game parameter  $\phi$ . This is meant to capture the process of financial development as the constraints in idiosyncratic insurance gets relaxed. We show that as entrepreneurs have access to a greater extent of idiosyncratic insurance, the idiosyncratic risk premium falls. The reduction in required returns stimulates entrepreneurs to expand their business, so the aggregate capital stock of the economy increases. Financial development goes together with economic development. Moreover, the reduction in the shadow price of idiosyncratic risk will lead to a reduction in within-group inequality. Second, we consider the effect of demographic transitions, where the population growth of the economy  $g$  is reduced. We show that, in contrast to partial equilibrium analysis that emphasize the role of the difference  $r - g$ , a reduction in the population growth *reduces* wealth inequality. The reason is the endogenous response of the idiosyncratic risk premium, which falls in response to a drop in  $g$ , leading to a reduction in inequality. Moreover, the capital stock of the economy increases, so a reduction in population growth has more benign effects than in environments where the role of idiosyncratic risk is ignored.

**Literature:** This paper is related to several strands of literature. First, the development literature on the risk and return of production activities: see, for instance, [Rosenzweig and Binswanger \(1992\)](#), [Morduch \(1995\)](#), [Udry and Anagol \(2006\)](#), [De Mel et al. \(2008\)](#), [Karlan et al. \(2014\)](#), [Beaman et al. \(2015\)](#). The literature on financial frictions and misallocation, as in [Buera and Shin \(2013\)](#), [Midrigan and Xu \(2014\)](#), [Moll \(2014\)](#). However, in contrast to most of this work, we focus on insurance constraints instead of borrowing constraints. The paper is also related to the literature on portfolio choice and asset pricing over the life cycle: [Poterba and Samwick \(1997\)](#), [Storesletten et al. \(2004\)](#), [Viceira \(2001\)](#), [Constantinides et al. \(2002\)](#), and [Cocco et al. \(2005\)](#).

The rest of the paper is organized as follows. Section 2 discuss the entrepreneur's problem and the risk-taking and savings decisions of entrepreneurs over the life cycle. Section 3 discuss the determination of between-group and within-group inequality. Section 4 closes the model in general equilibrium and discuss the equilibrium pricing conditions. Section 5 presents the financial development and demographic transition counterfactuals and section 6 concludes.

## 2 Risk-taking and savings over the life cycle

In this section, we will focus on the problem of an entrepreneur in isolation. We will focus on how consumption and risk-taking of entrepreneurs evolve over the life cycle, for a given sequence of prices. In particular, we will focus on prices consistent with a stationary equilibrium, where properly scaled prices are constants. In the next section, we will embed the entrepreneur's problem in a general equilibrium setting, and consider the determination of a stationary equilibrium.

## 2.1 Entrepreneur's problem

Consider the problem of an entrepreneur who lives for  $T$  periods and must choose at each instant how much to consume and how to divide her savings between a riskless financial investment and a risky production activity. The production technology is subject to both aggregate and idiosyncratic shocks. Entrepreneurs can purchase any amount of aggregate insurance, but they are limited on the amount of idiosyncratic insurance they can buy. We show in the appendix that this particular market structure implements the allocation under an optimal contract of a dynamic moral hazard problem.

### Production Technology

Entrepreneur  $i$  combines capital  $k_{i,t}$  and labor  $l_{i,t}$  to produce a final good  $y_{i,t}$  using the technology:

$$y_{i,t} = A_t k_{i,t}^\alpha l_{i,t}^{1-\alpha} \quad (1)$$

The entrepreneur is subject to two types of shocks. First, an aggregate productivity shock which follows a geometric Brownian motion:<sup>4</sup>

$$\frac{dA_t}{A_t} = \mu_A dt + \sigma_A dZ_t \quad (2)$$

where  $Z_t$  is a standard Brownian motion.

Capital accumulation is subject to *idiosyncratic* shocks:

$$\frac{dk_{i,t}}{k_{i,t}} = (l_{i,t} - \delta) dt + \sigma_{id} dZ_{i,t} \quad (3)$$

where  $l_{i,t}$  represents the net investment rate and is a choice variable to the entrepreneur.

Entrepreneurs can hire labor at the wage rate  $w_t$  and buy capital at the price  $q_t$ . We will focus on a stationary equilibrium, such that  $w_t = w^* A_t$  and  $q_t = q^* A_t$ . The relative price of capital then evolves according to  $dq_t/q_t = \mu_A dt + \sigma_A dZ_t$ . Investment is subject to quadratic adjustment costs  $\Phi(l_{i,t}) A_t k_{i,t}$ , where  $\Phi(l) = \bar{\Phi}_0 l + \frac{\bar{\Phi}_1}{2} l^2$ ,  $\bar{\Phi}_1 > 0$ . Adjustment costs depend on the amount of capital in *efficiency units*. This will be important to guarantee the economy has a balanced growth path.

The return of investing in the project can be written as

$$\begin{aligned} dR_{i,t} &= \frac{y_{i,t} - w_t l_{i,t} - \Phi(l_{i,t}) A_t k_{i,t}}{q_t k_{i,t}} dt + \frac{d(q_t k_{i,t})}{q_t k_{i,t}} \\ &= \mu_{i,t}^R dt + \sigma_A dZ_t + \sigma_{id} dZ_{i,t} \end{aligned}$$

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<sup>4</sup>The level of productivity can be written as  $A_t = A_0 e^{\left(\mu_A - \frac{\sigma_A^2}{2}\right)t + \sigma_A Z_t}$ , where  $Z_t \sim \mathcal{N}(0, t)$ , so productivity at any given date is distributed as a log-normal random variable.

where

$$\mu_{i,t}^R \equiv \underbrace{\frac{y_{i,t} - w_t l_{i,t} - \Phi(l_{i,t}) A_t k_{i,t}}{q_t k_{i,t}}}_{\text{dividend yield}} + \underbrace{\frac{\mu_A + l_{i,t} - \delta}{\text{expected capital gain}}}_{\text{expected capital gain}} \quad (4)$$

### Preferences and labor supply

Entrepreneurs have CRRA preferences with parameter  $\gamma$ , they live for  $T$  periods and derive utility of leaving bequests

$$U_{i,s_i} = \mathbb{E}_{s_i} \left[ \int_{s_i}^{s_i+T} e^{-\rho(t-s_i)} \frac{c_{i,t}^{1-\gamma}}{1-\gamma} dt + e^{-\rho(T-s_i)} (1-\psi)^\gamma V^* \frac{n_{i,s_i+T}^{1-\gamma}}{1-\gamma} \right] \quad (5)$$

where  $s_i$  is the date entrepreneur  $i$  was born and  $n_{i,t}$  denotes financial wealth (or net worth).

The parameter  $\psi$  measures the strength of the bequest motive. If  $\psi = 1$ , the entrepreneur gives no weight to future generations and there are no intergenerational linkages. If  $\psi = 0$  the behavior of the entrepreneur will coincide with the one of an agent with infinite horizon.<sup>5</sup> The case  $0 < \psi < 1$  then captures a form of imperfect altruism.

In addition to business income, entrepreneurs are allowed to receive labor income. Hence, as in the data, households have multiple sources of income. Labor is supplied inelastically, it is denoted by  $\bar{l}_{i,t}$ , and can vary (deterministically) over the life cycle.<sup>6</sup> We will show in section 2.3 that there is significant variation in the importance of labor income over the life cycle. The profile of labor supply over the life cycle will follow a flexible functional form in order to capture the corresponding empirical pattern:

$$\bar{l}_{i,t} = \Gamma_1 e^{\varphi_1(t-s_i)} + \Gamma_2 e^{\varphi_2(t-s_i)} \quad (6)$$

The sum of exponentials is analytically convenient and, as we are going to see, it fits well the empirical labor income life-cycle profile.

### The maximization problem

Entrepreneurs have access to a riskless savings instrument with return  $r_t$ . The entrepreneur can also buy aggregate and idiosyncratic insurance. The entrepreneur pays  $p_t^{ag} \theta_{i,t}^{ag}$  to reduce aggregate volatility by  $\theta_{i,t}^{ag}$ . There is no restriction in how much the entrepreneur can buy of aggregate insurance. In contrast, the cost of idiosyncratic insurance will be zero in equilibrium, as providers of insurance can perfectly diversify across entrepreneurs, but entrepreneurs can insure at most a fraction  $1 - \phi$  of the idiosyncratic volatility:

$$\theta_{i,t}^{id} \leq (1 - \phi) q_t k_{i,t} \sigma_{id} \quad (7)$$

<sup>5</sup>The coefficient  $V^*$  equals the value function coefficient for an infinite horizon agent.

<sup>6</sup>Notice that  $\bar{l}_{i,t}$  denotes the amount of labor supplied by an entrepreneur born at  $s$ , while  $l_{i,t}$  is the amount of labor demanded by an entrepreneur born at  $s$  to run his project.

where  $\theta_{i,t}^{id}$  is the amount of idiosyncratic insurance.

This *skin-in-the-game* constraint can be motivated by a moral hazard problem. If idiosyncratic shocks are private information and entrepreneurs can divert at least part of the capital, then entrepreneurs will have to bear some of the risk in order to support some risk-sharing. In the appendix, we derive the skin-in-the-game constraint following the literature on dynamic moral hazard problems.

Entrepreneurs in this economy are *insurance-constrained*, as they are forced to hold idiosyncratic risk if they decide to operate their production technology. In contrast, we abstract from (ad-hoc) borrowing constraints. First, this will allow us to highlight the implications of the less explored insurance constraint. Second, [Samphantharak and Townsend \(2018\)](#) shows that the negative correlation between wealth and returns, an indicative of borrowing constraint, disappears after we control for the appropriate risk premium. This is suggestive that, at least in the context of Thailand during this period, imperfect risk sharing may be a more prominent friction. Entrepreneurs can then borrow freely against their human wealth, i.e., the value of their future labor income:

$$n_{i,t} \geq -h_{i,t} \quad (8)$$

where  $h_{i,t}$  denotes human wealth.<sup>7</sup>

The entrepreneur's problem is to choose a vector of processes  $(c_i, k_i, l_i, \bar{l}_i, \theta_i^{ag}, \theta_i^{id})$ , taking the processes for prices  $(q, w, r, p^{ag})$  as given, to solve the following program:

$$V_{i,s_i} = \max_{c_i, k_i, l_i, \bar{l}_i, \theta_i^{ag}, \theta_i^{id}} \mathbb{E}_{s_i} \left[ \int_{s_i}^{s_i+T} e^{-\rho(t-s_i)} \frac{c_{i,t}^{1-\gamma}}{1-\gamma} dt + e^{-\rho(T-s_i)} (1-\psi)^\gamma V^* \frac{n_{i,s_i+T}^{1-\gamma}}{1-\gamma} \right] \quad (9)$$

subject to (7), (8), non-negativity constraints  $c_{i,t}, k_{i,t} \geq 0$ , and the law of motion of  $a_{i,t}$

$$dn_{i,t} = \left[ (n_{i,t} - q_t k_{i,t}) r_t + q_t k_{i,t} \mu_{i,t}^R - p_t^{ag} \theta_{i,t}^{ag} + w_t \bar{l}_{i,t} - c_{i,t} \right] dt + \left( q_t k_{i,t} \sigma_A - \theta_{i,t}^{ag} \right) dZ_t + \left( q_t k_{i,t} \sigma_{id} - \theta_{i,t}^{id} \right) dZ_{i,t}$$

given  $n_{i,s_i} > -h_{i,s_i}$ .

The term in brackets in the expression above represents the expected growth rate of financial wealth. The entrepreneur invests  $n_{i,t} - q_t k_{i,t}$  in the riskless asset, with return  $r_t$ , and the amount  $q_t k_{i,t}$  in the risky technology, with expected return  $\mu_{i,t}^R$ . The cost of aggregate insurance is  $p_t^{ag} \theta_{i,t}^{ag}$ . The entrepreneur receives labor income  $w_t \bar{l}_{i,t}$  and consumes  $c_{i,t}$ . The last two terms represent the exposure to aggregate and idiosyncratic risk. The risk exposure depends on the scale on which the business is operated,  $q_t k_{i,t}$ , volatilities,  $\sigma_A$  and  $\sigma_{id}$ , and aggregate and idiosyncratic insurance,  $\theta_{i,t}^{ag}$  and  $\theta_{i,t}^{id}$ .

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<sup>7</sup>Human wealth is given by the discounted value of future labor income  $h_{i,t} \equiv \mathbb{E}_t \left[ \int_t^{s+T} \frac{\pi_z}{\pi_t} w_z \bar{l}_{i,z} dz \right]$ , where  $\frac{\pi_z}{\pi_t}$  denotes the stochastic discount factor for this economy.

## 2.2 Solution in a stationary equilibrium

We will focus on a *stationary equilibrium* where, despite the aggregate shocks, scaled variables are constant. This is the analogous of a balanced growth path to a stochastic economy. The economy is in a stationary equilibrium if prices satisfy  $w_t = wA_t$ ,  $q_t = qA_t$ ,  $r_t = r$ ,  $p_t^{ag} = p^{ag}$  and the aggregate capital-labor ratio is constant and denoted by  $k$ . We will maintain this assumption through this section.

### Maximizing expected returns

Notice that  $(l_{i,t}, k_{i,t})$  only enter the maximization problem through  $\mu_{i,t}^R$ . Hence, the entrepreneur will choose these variables to maximize the expected return on the business. Labor demand assumes the usual form:

$$w = (1 - \alpha) \left( \frac{k_{i,t}}{l_{i,t}} \right)^\alpha \quad (10)$$

Hence, the capital-labor ratio is equalized across entrepreneurs and it will equal the aggregate capital-labor ratio:  $k_{i,t}/l_{i,t} = k$ .

The net investment rate  $\iota_{i,t}$  is given by

$$\Phi'(\iota_{i,t}) = q \Rightarrow \iota_{i,t} = \frac{q - \bar{\Phi}_0}{\bar{\Phi}_1} \equiv \iota(q) \quad (11)$$

using the quadratic cost specification  $\Phi(\iota) = \bar{\Phi}_0\iota + 0.5\bar{\Phi}_1\iota^2$ .

Since the capital-labor ratio and the net investment rate are all equalized across entrepreneurs, the expected return on the business is also equalized. We will then denote the expected return by  $\mu_{i,t}^R = \mu^R$ . Plugging the previous two equations into (4), we obtain

$$\mu^R = \frac{\alpha k^{\alpha-1} - \Phi(\iota(q))}{q} + \mu_A + \iota(q) - \delta \quad (12)$$

### Financial and human wealth

Lemma 1 indicates that the relevant notion of wealth for the entrepreneur is *total wealth*,  $\omega_{i,t} = n_{i,t} + h_{i,t}$ , the sum of financial and human wealth.

**Lemma 1.** *Suppose the economy is in a stationary equilibrium.*

(a) *The value function is given by*

$$V_{i,t}(n_{i,t}) = V_{s_{i,t}}^* \frac{(n_{i,t} + h_{i,t})^{1-\gamma}}{1-\gamma} \quad (13)$$

where  $V_{s_{i,t}}^*$  is deterministic and given in the appendix.

The effective risk aversion of entrepreneur  $i$  is given by

$$-\frac{V_{i,nn}n_{i,t}}{V_{i,n}} = \frac{\gamma}{1 + \frac{h_{i,t}}{n_{i,t}}} \quad (14)$$



(b) Human wealth evolves according to

$$dh_{i,t} = \left[ (r + p^{ag}\sigma_A) h_{i,t} - w_t \bar{l}_{i,t} \right] dt + h_{i,t} \sigma_A dZ_t \quad (15)$$

Human wealth is then given by

$$h_{i,t} = \int_t^{s_i+T} e^{-(r+p^{ag}\sigma_A)(z-t)} \mathbb{E}_t[w_{t+z}] \bar{l}_{i,z} dz \quad (16)$$

(c) Demand for capital, aggregate, and idiosyncratic insurance solve the mean-variance problem:

$$\max_{k_i, \theta_i^{ag}, \theta_i^{id}} \left\{ \frac{q_t k_{i,t}}{n_{i,t}} (\mu^R - r) + \frac{h_{i,t}}{n_{i,t}} p^{ag} \sigma_A - p^{ag} \frac{\theta_{i,t}^{ag}}{n_{i,t}} - \frac{1}{2} \frac{\gamma}{1 + \frac{h_{i,t}}{n_{i,t}}} \left[ \left( \frac{q_t k_{i,t}}{n_{i,t}} \sigma_A + \frac{h_{i,t}}{n_{i,t}} \sigma_A - \frac{\theta_{i,t}^{ag}}{n_{i,t}} \right)^2 + \left( \frac{q_t k_{i,t}}{n_{i,t}} \sigma_{id} - \frac{\theta_{i,t}^{id}}{n_{i,t}} \right)^2 \right] \right\} \quad (17)$$

subject to (7).

The first part of lemma 1 gives the value function, an age-dependent CRRA function of total wealth. Importantly, the effective (value function) risk aversion of an entrepreneur will vary depending on the human-to-financial wealth ratio. As discussed in section 2.3, this ratio varies significantly over the life cycle in the Thai data, and it will play an important role explaining the differences in risk-taking over the life cycle.

The second part of the lemma gives the law motion of human wealth. Human wealth is the present discounted value of future labor income. Implicitly, this expression gives the appropriate discount rate to compute human wealth. Since the wage will move with aggregate productivity, labor income is risky, and the discount rate for human wealth incorporates the risk premium  $p^{ag}\sigma_A$ .

The final part of the lemma shows that the portfolio choice of an entrepreneur reduces to a simple mean-variance problem with risk aversion  $\gamma/(1 + \frac{h_{i,t}}{n_{i,t}})$ . This is the result of the continuous-time formulation, as the mean-variance objective comes from a direct rearrangement of the HJB equation. The first term in (17) captures the expected excess return on total wealth, which comes from the return on the business and from human wealth. The second term is the product of the effective risk aversion and the aggregate and idiosyncratic variance. The maximization problem is subject to the skin-in-the-game constraint. The Lagrange multiplier to this constraint, which we refer to as the *shadow price of idiosyncratic insurance*, will play an important role in the characterization of the entrepreneur's risk-taking decision.

### Savings behavior and risk-taking over the life cycle

The next proposition characterizes the savings and risk-taking decisions of entrepreneurs

**Proposition 1.** *Suppose the economy is in a stationary equilibrium.*

(a) *The shadow price of idiosyncratic insurance is*

$$p^{id} = \frac{\mu^R - r - p^{ag}\sigma_A}{\phi\sigma_{id}} \quad (18)$$

(b) Demand for capital is given by

$$\frac{q_t k_{i,t}}{n_{i,t}} = \frac{1 + \frac{h_{i,t}}{n_{i,t}} p^{id}}{\gamma \phi \sigma_{id}} \quad (19)$$

(c) The demand for aggregate insurance is

$$\frac{\theta_{i,t}^{ag}}{n_{i,t}} = \left[ \frac{1 + \frac{h_{i,t}}{n_{i,t}} p^{id}}{\gamma \phi \sigma_{id}} + \frac{h_{i,t}}{n_{i,t}} \right] \sigma_A - \frac{1 + \frac{h_{i,t}}{n_{i,t}} p^{ag}}{\gamma} \quad (20)$$

(d) The consumption-wealth ratio is given by

$$\frac{c_{i,t}}{n_{i,t}} = \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-(t-s_i))}} \left( 1 + \frac{h_{i,t}}{n_{i,t}} \right) \quad (21)$$

$$\text{where } \bar{r} = \frac{1}{\gamma} \rho + \left( 1 - \frac{1}{\gamma} \right) r^{MV} \text{ and } r^{MV} \equiv r + \frac{(p^{ag})^2 + (p^{id})^2}{2\gamma}.$$

The first part of the proposition says the shadow price of idiosyncratic insurance, the Lagrange multiplier on the skin-in-the-game constraint, is equalized across entrepreneurs. Moreover, it equals the return per unit of risk (Sharpe ratio) of an investor who fully insures the project against aggregate risk, so it is exposed only to idiosyncratic risk. In equilibrium, this term will be positive and the entrepreneur will buy as much idiosyncratic insurance as possible, i.e.,  $\theta_{i,t}^{id} = (1 - \phi) \sigma_{id}$ .

Demand for capital depends on the effective risk aversion and the price and quantity of *idiosyncratic* risk. Cross-sectional differences in risk-taking are captured by differences in the effective risk aversion, in particular, the human-to-financial wealth ratio. Since this ratio has an important life-cycle component, risk-taking will also vary over the life-cycle. The average scale of the business will depend on the ratio of the shadow price of idiosyncratic risk and the amount of idiosyncratic volatility the entrepreneur cannot insure. Hence, the decision about the scale of the business does not depend on aggregate risk. The reason is the possibility of sharing risk through the insurance contract. An entrepreneur that is relatively less risk averse can increase the scale of the business and get rid of the additional aggregate risk by buying insurance. The decision of how much capital to have in the business is essentially a decision of how much idiosyncratic risk to hold. In particular, if there is no idiosyncratic risk or if the entrepreneur can insure all of it,  $\phi = 0$ , then the decision of how much to invest in the business is indeterminate: the entrepreneur is indifferent between investing in the safe asset or in the business, even if aggregate risk is still present.

Equation (20) gives the demand for aggregate insurance. The demand is linear in the price of aggregate insurance with a slope given by the inverse of the effective risk aversion and intercept given by the total exposure to aggregate risk, coming from the business and from human wealth.

Finally, the expression for the consumption-financial-wealth ratio is given in (21). The first term  $\bar{r}/(1 - \psi e^{-\bar{r}(T-(t-s_i))})$  is the *marginal propensity to consume* (MPC). It is increasing in age, as it is typical in finite horizon problems, and the bequest motive parameter  $\psi$  controls the strength of this effect. If  $\psi = 0$ , the MPC will be constant, recovering the result of the infinite horizon problem. If  $\psi = 1$ , then the MPC will get arbitrarily large as the entrepreneur approaches the end of life, so all the stock of wealth

will be consumed at the final age  $T$ . While  $\psi$  is important to determine how the MPC varies over the life cycle,  $\bar{r}$  is important to determine the average MPC. If  $\gamma = 1$ , then  $\bar{r} = \rho$ , the entrepreneur's discount rate. In general,  $\bar{r}$  is a linear combination of  $\rho$  and  $r^{MV}$ , the variance adjusted expected return on total wealth  $\omega_{i,t}$ . The term  $r^{MV}$  can be written as a mean-variance objective  $r^{MV} = \frac{1}{dt} \left[ \mathbb{E} \left[ \frac{d\omega_{i,t}}{\omega_{i,t}} \right] - \frac{\gamma}{2} \mathbb{V} \left[ \frac{d\omega_{i,t}}{\omega_{i,t}} \right] \right]$ . After some simplification, we obtain  $r^{MV} = r + \frac{(p^{ag})^2 + (p^{id})^2}{2\gamma}$ . Hence, if  $\gamma > 1$ , an increase in risk-adjusted returns will increase the average MPC.<sup>8</sup> Finally, the consumption-wealth ratio depends on the human-financial wealth ratio, which potentially vary over the life cycle. Entrepreneurs with more human wealth will consume more out of their current assets.

### 2.3 Testing the life cycle predictions

The human-financial wealth ratio plays an important role on how risk-taking and consumption decisions vary over the life cycle. In particular, differences in risk-taking over the life cycle are entirely determined by differences in the human-financial wealth ratio. Hence, the ability of the theory to make testable predictions rely on the ability to discipline  $h_{i,t}/n_{i,t}$  empirically. From equation (16), this requires a measure of expected returns,  $r + p^{ag}\sigma$ , and a measure of how expected labor income varies over the life cycle,  $\mathbb{E}_t[w_{t+z}\bar{l}_{i,z}]$ . We turn next to the description of the data and the measurement of these variables.

#### Data

We use data from the Townsend Thai Monthly Survey, an ongoing intensive monthly survey initiated in 1998 in four provinces of Thailand. Two provinces, Chachoengsao and Lopburi, are semi-urban in a more developed central region near the capital, Bangkok. The other two provinces are rural, Buriram and Srisaket, and are located in the less developed northeastern region by the border of Cambodia. In each of the four provinces, the survey is conducted in four villages, chosen at random within a given township.<sup>9</sup>

Our sample covers 716 households and 14 years of monthly data, starting in January 1999. During this time, these village economies were subject to all sorts of aggregate and idiosyncratic shocks. Rice cultivation is affected by seasonal variation in rainfall and temperature. Restrictions on exports to the EU affected shrimp ponds. The productivity of milk cows varies substantially both over time for a given animal and over the heard.

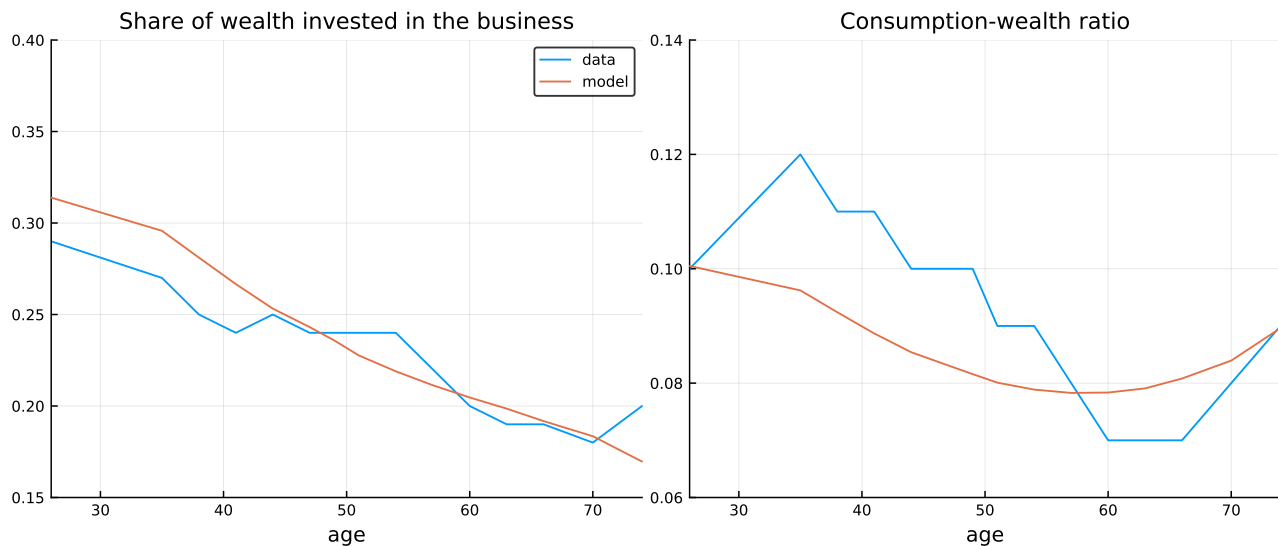
The data collected in the Townsend Thai Monthly Survey includes information on the net income generated by the business as well as total assets and liabilities of households. It also includes information on the household's labor income and consumption. The construction of these variables is described in detail in [Samphantharak and Townsend \(2018\)](#). Figure 1 shows the life cycle profile for the share of financial wealth invested in the business and the consumption-wealth ratio. The share of wealth invested in the business falls sharply over the life cycle. From around 30% of financial wealth

<sup>8</sup>This is an extension of the usual result that interest rates have income and substitution effect on savings decisions, where the income effect dominates for  $\gamma > 1$ . In an environment with risky returns, the relevant notion is  $r^{MV}$  instead of the riskless interest rate.

<sup>9</sup>For details on the Townsend Thai Monthly Survey, see [Samphantharak and Townsend \(2010\)](#).

at age 25, to less than 20% at the end of cycle, a drop of more than 40%. The consumption-wealth ratio is roughly U-shaped over the life cycle. It declines until the sixties, then it returns back up. Again, there is large variation over the life cycle. From peak to trough, the consumption-wealth ratio falls by more than 40%.

**Figure 1: Risk-taking and Savings - Life Cycle Profiles**



### Measurement of returns and volatilities

The return on the business is measured as the net income divided by total assets net of liabilities, i.e., the return on assets (ROA), a conventional financial accounting measure of performance of productive assets. Given our measure of returns, we can compute the expected return and volatility for aggregate and idiosyncratic risk as follows. First, define the “market return” as the return on a diversified portfolio of individual businesses:

$$dR_{M,t} = \mu_M dt + \sigma_A dZ_t$$

Notice that the market portfolio is not exposed to idiosyncratic risk, only aggregate risk. When the market portfolio is publicly traded, then no-arbitrage imply  $\mu_M = r + p^{ag}\sigma_A$ . We can now use the market return to obtain an estimate of the shadow price of idiosyncratic risk. Taking the difference between the return on individual business  $i$  and the market portfolio and rearranging

$$\begin{aligned} dR_{i,t} &= [\mu_R - r - p^{ag}\sigma_A] dt + dR_{M,t} + \sigma_{id}dZ_{i,t} \\ &= \tilde{\alpha}_i dt + \tilde{\beta}_i dR_{M,t} + \sigma_{id}dZ_{i,t} \end{aligned}$$

where  $\tilde{\alpha}_i = p^{id}\phi\sigma_{id}$  and  $\tilde{\beta}_i = 1$ .

The terms  $\tilde{\alpha}_i$  and  $\tilde{\beta}_i$  would be the intercept and slope, respectively, of a time-series regression of individual returns on the market (average) return.<sup>10</sup> If the market portfolio is not directly traded, then

<sup>10</sup>Under our simplifying assumptions of no differences in productivity or exposure to aggregate risk, the coefficients of the regression would be the same across entrepreneurs. In general, differences in productivity or risk would lead to differences

$\mu_M$  would include an idiosyncratic risk premium and then identification would require an additional step: regressing expected returns on betas, as in Fama and MacBeth (1973). The coefficient on the beta would recover the price of aggregate risk and the slope would give the price of idiosyncratic risk. The idiosyncratic volatility can be identified from the volatility of the residuals of this regression. Since the variance of individual returns is  $\sigma_A^2 + \sigma_{id}^2$ , we can derive  $\sigma_A$  from the volatility of returns and  $\sigma_{id}$ . Finally, the price of aggregate insurance can be identified from the expected returns on the market portfolio, which can be identified by a time-series average of the market return. Given the parameter  $\phi$  that controls the extent of idiosyncratic insurance, the shadow price of idiosyncratic insurance can be obtained from  $\tilde{\alpha}_i$ . We will take  $\phi$  as given for now and discuss its identification in a later section after covering the determination of equilibrium prices.

These return regressions, and several variations of them, were extensively studied in [Samphantharak and Townsend \(2018\)](#). We can then obtain estimates for the aggregate and idiosyncratic volatilities and returns directly from their work.<sup>11</sup>

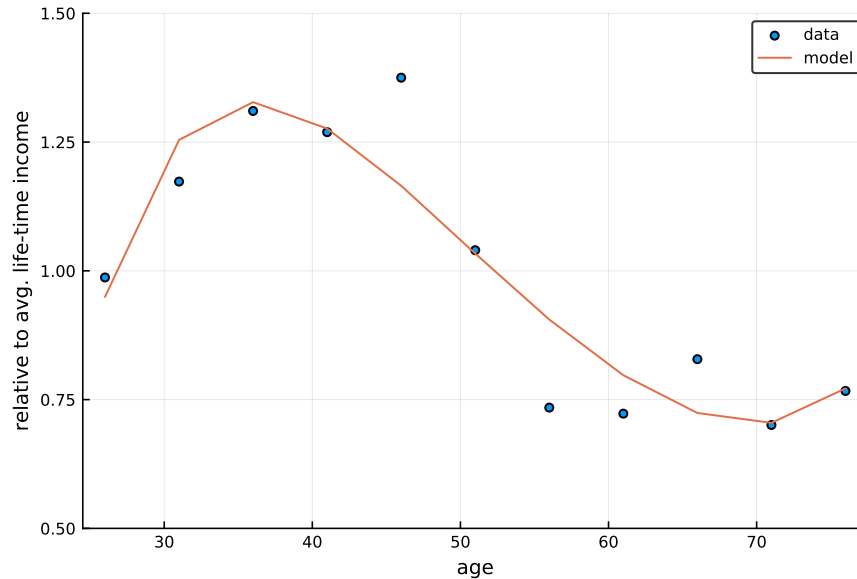
### Measurement of human wealth

From (16), human wealth can be computed as

$$h_{i,t} = \int_t^{s_i+T} e^{-(r+p^{ag}\sigma_A-\mu_A)(z-t)} w_i \bar{l}_e \left[ \Gamma_1 e^{\phi_1(z-s_i)} + \Gamma_2 e^{\phi_2(z-s_i)} \right] dz \quad (22)$$

using the fact that  $\mathbb{E}_t[w_z] = e^{\mu_A(z-t)} w_t$  and  $\bar{l}_{i,z} = \Gamma_1 e^{\phi_1(z-s_i)} + \Gamma_2 e^{\phi_2(z-s_i)}$ .

**Figure 2:** Labor Income - Life Cycle Profile



The term  $r + p^{ag}\sigma_A$  was obtained from the return on the market portfolio and  $\mu_A$  is simply the aggregate per capita growth rate of the economy. Notice that if  $T \rightarrow \infty$  and  $\bar{l}_{i,z}$  was constant, then the

in  $\tilde{\alpha}_i$  and  $\tilde{\beta}_i$ .

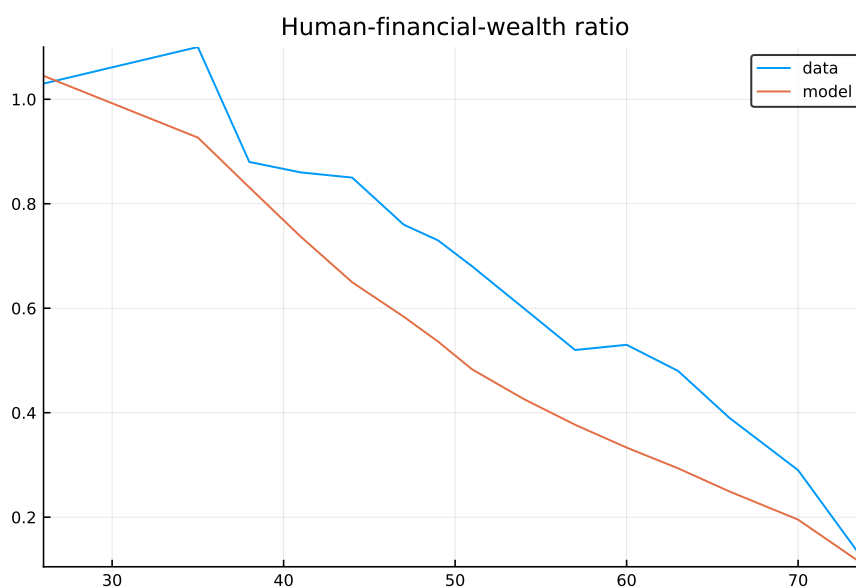
<sup>11</sup>[Samphantharak and Townsend \(2018\)](#) consider the extremes of full risk sharing,  $\phi = 0$  in our notation, and autarky,  $\phi = 1$ , but they do not consider explicitly the intermediary cases presented by  $0 < \phi < 1$

expression above would boil down to the Gordon growth formula:  $h_{i,t} = w_t \bar{l}_e / (r + p^{ag} \sigma_A - \mu_A)$ .<sup>12</sup>

The term  $w_t \bar{l}_e$  represents the average labor income of entrepreneurs across all age groups. Hence,  $\sum_{j=1}^2 \Gamma_j e^{\varphi_j(z-s_i)}$  represents the labor income of entrepreneurs with age  $z - s_i$  relative to the average. Using data on the relative labor income of each age group, we estimate the parameters  $(\Gamma_j, \varphi_j)$  by non-linear least squares. Figure 2 shows that the functional form does a good job of approximating the empirical labor income profile.

Given the discount rate and the labor income profile, we can compute the human-financial wealth, both in the data and in the model. Except for early in life, the human-financial wealth ratio tend to decline over the life cycle, as reported in figure 3. Human wealth is quantitatively as important as financial wealth at the beginning of the cycle. By the age of 50, human wealth is only half of the financial wealth. The model does reasonably well in capturing the evolution of the human-financial wealth over the life cycle, even though it does not capture the initial increase in human wealth. The parameter  $\bar{l}_e$  is calibrated to match the average value of  $h_{i,t}/n_{i,t}$  across age groups, and the evolution over the life is determined by the discount rate and the labor income profile estimated above.

**Figure 3: Human-financial wealth ratio - Life Cycle Profile**



### Implications for risk-taking and savings

In order to solve for  $q_t k_t / n_{i,t}$  in the model, it remains to calibrate the risk aversion parameter  $\gamma$ . We will set  $\gamma = 6.7$ , such that the model implied aggregate risk premium  $p^{ag} \sigma_A$  coincides with the estimate from the return regressions discussed above. Given the decline in the human-financial wealth ratio, the effective risk aversion of the entrepreneur increases over the life cycle, leading to a reduction in the share of wealth invested in the risky business, as shown in figure 1. Notice the ratio between the share invested in the business at the beginning and the end of life is entirely determined by the human-

<sup>12</sup>The Gordon growth formula is an expression typically used to value stocks which says the value of a company equals the dividend divided by the difference between the investor's discount rate and the growth rate of dividends.

financial wealth ratio, which was calibrated independently of information on the cross-sectional of risk-taking.

To compute the consumption-wealth ratio, it remains to calibrate  $\rho$  and  $\psi$ . As previously discussed, the average consumption-wealth ratio is particularly informative about  $\bar{r}$ , and so  $\rho$ , and the ratio of the consumption-wealth ratio at the beginning and the end of life is informative about  $\psi$ . We will use these moments to calibrate  $(\rho, \psi)$ . The model is able to replicate the U-shaped pattern in  $c_{i,t}/n_{i,t}$  over the life cycle, although with less curvature. This is the result of two opposing forces. On the one hand, the decline in the human-financial wealth ratio tends to reduce  $c_{i,t}/n_{i,t}$ . On the other hand, the increase in the MPC tends to increase the consumption-wealth ratio. This force is particularly strong by the end of life, overturning the impact of the reduction in  $h_{i,t}/n_{i,t}$ .

### 3 Inequality between and within age groups

The presence of uninsurable idiosyncratic risk will distort the distribution of wealth in the economy. With perfect insurance, there is no wealth inequality within age groups, and wealth across age groups will only vary for standard life cycle considerations. Imperfect insurance will create dispersion in wealth within age groups, and wealth across age groups will depend on risk and risk premium.

In this section, we will characterize how the wealth distribution is determined by the interaction of risk and demographics. First, we will characterize how wealth evolves on average over the life cycle. Second, we will show how we can solve for the wealth distribution within age groups. In both cases, the risk-taking and savings decisions discussed in section 2 will play an important role.

**Demographics and notation:** First, let's detail the demographic structure of the economy and fix some notation. Population grows at rate  $g$ . In particular, the index  $i$  of an entrepreneur born at date  $s_i = s$  belongs to the set  $[0, e^{gs}]$ . Hence, the mass of agents in cohort  $s$  is  $e^{gs}$  and total population at date  $t$  is  $e^{gt}(1 - e^{-gT})/g$ . The child of entrepreneur  $i$  will have index  $ie^{gT}$  and it will be born at date  $s_i + T$ . The child will inherit the financial wealth of the parent, so  $a_{ie^{gT}, s_i + T} = a_{i, s_i + T}$ . The age distribution in the population is a truncated exponential with density

$$f(a) = \frac{ge^{-ga}}{1 - e^{-gT}} \quad (23)$$

Let  $x_{s,t} = e^{-gs} \int_{\{i:s_i=s\}} x_{i,t} di$  and  $x_{e,t} = \int_{t-T}^t f(t-s)x_{s,t} ds$  denote, respectively, the average value of variable  $x$  across entrepreneurs born at date  $s$  and across all entrepreneurs, for any variable  $x_{i,t}$ .

#### 3.1 Between-group inequality

Plugging in equations (19) and (20) into the law of motion of financial wealth, we obtain the expression

$$\frac{dn_{i,t}}{n_{i,t}} - \mathbb{E}_t \left[ \frac{dn_{i,t}}{n_{i,t}} \right] = \left( \frac{1 + \frac{h_{i,t}}{n_{i,t}}}{\gamma} p_t^{ag} - \frac{h_{i,t}}{n_{i,t}} \sigma_A \right) dZ_t + \frac{1 + \frac{h_{i,t}}{n_{i,t}}}{\gamma} p^{id} dZ_{i,t}$$

Notice that an increase in  $h_{i,t}/n_{i,t}$  has two opposing effects on the exposure to aggregate risk. It reduces the effective risk aversion, leading the entrepreneur to buy less aggregate insurance, but it

increases background risk, leading the entrepreneur to buy more insurance. The net effect will be zero if  $p^{as} = \gamma\sigma_A$ . This is a standard formula in asset pricing theory for the price of aggregate risk and it will hold in the stationary equilibrium in this economy. Using this expression for  $p^{as}$ , we find that entrepreneurs will have the same exposure to aggregate risk:

$$\frac{dn_{i,t}}{n_{i,t}} - \mathbb{E}_t \left[ \frac{dn_{i,t}}{n_{i,t}} \right] = \sigma_A dZ_t + \frac{1 + \frac{h_{i,t}}{n_{i,t}}}{\gamma} p^{id} dZ_{i,t}$$

An implication of this common exposure to aggregate risk is that financial and human wealth of entrepreneurs move in tandem with aggregate shocks. This property is important to guarantee a stationary distribution of wealth exists, despite the presence of aggregate risk. Consider  $n_{s,t}$  and  $n_{e,t}$ , the average financial wealth of entrepreneurs born at date  $s$  and the average across all entrepreneurs, respectively. Because we are averaging over a continuum of entrepreneurs, the idiosyncratic risk is completely diversified, but  $n_{s,t}$  still respond to aggregate risk. However, since  $n_{s,t}$  and  $n_{e,t}$  move with aggregate shocks by the same proportion, then the share of wealth held by entrepreneur with age  $a$ ,  $n_a = f(a)n_{t-a,t}/n_{e,t}$ , is non-stochastic and independent of calendar time  $t$ . Similarly, define  $h_a = f(a)h_{t-a,t}/n_{e,t}$ .

The next proposition provides a characterization of  $n_a$ .

**Proposition 2** (Between-group inequality). *Suppose the economy is in a stationary equilibrium. The share of wealth held by entrepreneurs of age  $a$ ,  $n_a$ , satisfies*

$$\log n_a = \log n_0 + \underbrace{\log \left( \frac{1 + \frac{h_0}{n_0}}{1 + \frac{h_a}{n_a}} \right)}_{\text{human-financial wealth effect}} + \underbrace{\left[ r + \frac{(p^{as})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) \right]}_{\text{generalized "r-g" effect}} a - \underbrace{\int_0^a \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-\bar{a})}} d\bar{a}}_{\text{average MPC effect}} \quad (24)$$

where

$$n_0 = \frac{e^{\left( r + \frac{(p^{as})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - mpc_e \right) T}}{1 - e^{\left( r + \frac{(p^{as})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - mpc_e \right) T}} h_0 \quad (25)$$

and  $mpc_e = \frac{1}{T} \int_0^T \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}} da$  is the average MPC across all entrepreneurs.

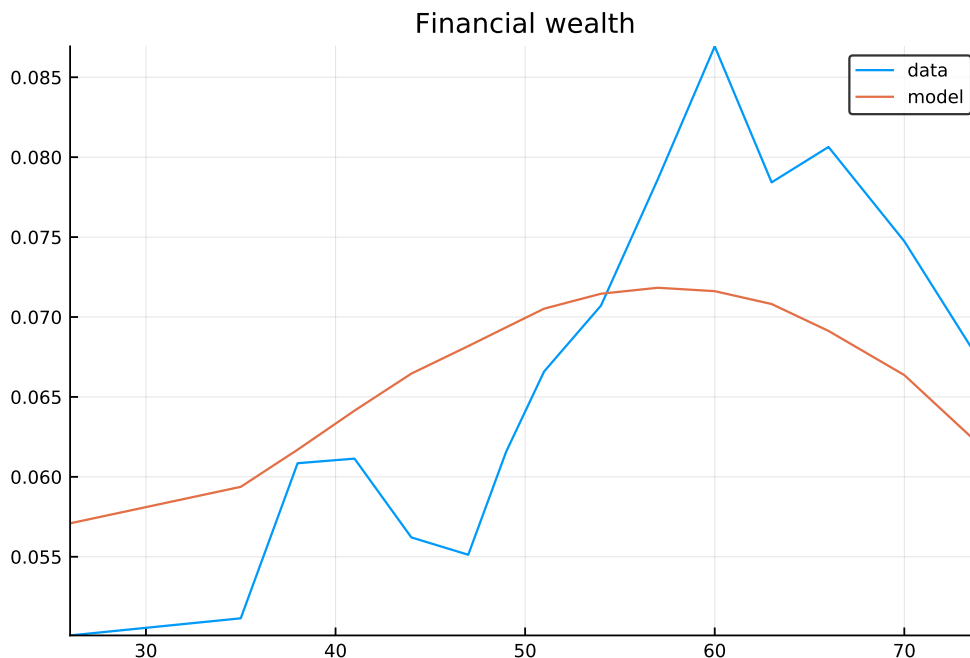
Proposition 2 decomposes the distribution of wealth across different age groups into three effects. First, a human-financial wealth effect. As the entrepreneur gets older, the human wealth gets "converted" into financial wealth, i.e., the labor income received accelerates the accumulation of financial wealth. The second term is a generalized "r - g" effect. The first component is the return on total wealth:  $r + (p^{as})^2/\gamma + (p^{id})^2/\gamma$ . In the absence of risk, this term simplifies to  $r$ . Hence, the correct notion of return in this context is the return on total wealth taking into account the aggregate and idiosyncratic risk premium. The second component is the growth rate of the economy,  $g + \mu_A$ , the sum of population and productivity growth. The generalized "r - g" effect implies that if the return on total wealth exceeds the growth rate of the economy, the share of wealth will tend to increase with age, as the entrepreneur has more time to accumulate wealth relative to the other entrepreneurs. The third term is the average MPC effect. It captures the fact that the amount of wealth an entrepreneur accumulates at age  $a$  depends on her previous consumption decisions. In particular, this term tends to



be low at the beginning of life and it increases as the entrepreneur gets older. Entrepreneur will then tend to accumulate wealth at the beginning of life, but eventually consumption will increase until the wealth achieves the desired bequest level.

The share of wealth held by the youngest households depends on the bequest of the older generation, so it depends on the cumulative return net of consumption over a lifetime. Hence, it is increasing in returns and decreasing in the growth rate of the economy and the average MPC. Financial wealth of the young also depend on the initial amount of human wealth.

**Figure 4:** Financial wealth distribution across age groups



Even though idiosyncratic risk is diversified by computing group averages, the presence of the idiosyncratic *risk premium* will affect between-group inequality through these three channels. Idiosyncratic risk premium increases the generalized " $r - g$ " term and it will affect the level and dispersion of MPCs across age groups as  $\bar{r}$  is a function of  $p^{id}$ . Moreover, since the idiosyncratic risk premium affects financial wealth accumulation, it will affect the evolution of the human-financial wealth ratio.

Figure 4 shows how financial wealth varies across age groups. Even though this was not part of the calibration targets, the model is able to capture the inverted U pattern of financial wealth. However, the variation in the model is more modest than the one in the data, potentially due to the smoother consumption-wealth generated by the model.

### 3.2 Within-group inequality

We have considered so far the behavior of the average wealth of entrepreneurs by age. However, even after conditioning on a given age group, wealth may vary substantially due to presence of idiosyncratic risk. We now turn to the characterization of the whole wealth distribution conditional on age.

In order to eliminate the impact of aggregate risk, consider the normalized wealth of agent  $i$ :  $\tilde{n}_{i,t} = n_{i,t}/n_{e,t}$ . Similarly, define the normalized human wealth  $\tilde{h}_{i,t} = h_{i,t}/n_{e,t}$ . Notice that  $\tilde{h}_{i,t}$  is a deterministic function of  $a_i = t - s_i$ , the age of entrepreneur  $i$ . Normalized wealth evolves according to

$$d\tilde{n}_{i,t} = \left[ \underbrace{\left( r + \frac{(p^{as})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - mpc_{a_i} - \mu_A \right)}_{\mu_{\tilde{n}}(\tilde{n}_{i,t}, a_i)} (\tilde{n}_{i,t} + \tilde{h}_{i,t}) - \underbrace{\left( r + \frac{(p^{as})^2}{\gamma} - \frac{w_i \bar{l}_{i,t}}{h_{i,t}} - \mu_A \right)}_{\sigma_{\tilde{n}}(\tilde{n}_{i,t}, a_i)} \tilde{h}_{i,t} \right] dt + \underbrace{\frac{p^{id}}{\gamma} (\tilde{n}_{i,t} + \tilde{h}_{i,t})}_{\sigma_{\tilde{n}}(\tilde{n}_{i,t}, a_i)} dZ_{i,t}$$

where  $mpc_a = \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}}$  and  $\frac{w_i \bar{l}_{i,t}}{h_{i,t}}$  is a deterministic function of  $a_i$ .

Notice that the expected change,  $\mu_{\tilde{n}}(\tilde{n}_{i,t}, a_i)$ , and the volatility,  $\sigma_{\tilde{n}}(\tilde{n}_{i,t}, a_i)$ , of normalized wealth depend only of the current value of  $\tilde{n}_{i,t}$  and the age of the entrepreneur  $a_i$ . We will denote the joint density of normalized financial wealth and age by  $f(\tilde{n}, a)$  and the conditional density by  $f(\tilde{n}|a)$ , already imposing the fact that the distribution does not depend on calendar time in a stationary equilibrium.

**Lemma 2** (Kolmogorov Forward Equation). *The conditional distribution of normalized wealth  $f(\tilde{n}|a)$  satisfies the partial differential equation*

$$\frac{\partial f(\tilde{n}|a)}{\partial a} = -\frac{\partial [f(\tilde{n}|a)\mu_{\tilde{n}}(\tilde{n}, a)]}{\partial \tilde{n}} + \frac{1}{2} \frac{\partial^2 [f(\tilde{n}|a)\sigma_{\tilde{n}}^2(\tilde{n}, a)]}{\partial \tilde{n}^2} \quad (26)$$

and the boundary condition  $f(\tilde{n}|0) = f(\tilde{n}|T)$ .

Despite the complexity created by the age-dependent expected change and volatility of wealth, we are able to solve for the conditional distribution of financial wealth in closed-form for the special case where entrepreneurs leave no bequests, i.e.,  $\psi = 1$ . This allow us to characterize analitically the evolution of inequality over the life cycle.

**Proposition 3** (Within-group inequality: no bequests). *Suppose  $\psi = 1$  and  $r + \frac{(p^{as})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A > 0$ .*

i. **Shifted log-normal distribution:** *The distribution of normalized wealth  $\tilde{n}_{i,t}$  conditional on age  $a_i = a$  is given by a shifted log-normal distribution with support  $(-\tilde{h}_a, \infty)$ , i.e., total wealth  $\tilde{n}_{i,t} + \tilde{h}_{i,t}$  is log-normally distributed.*

ii. **Mean and variance by age:** *The expected value and variance of  $\tilde{n}_{i,t}$  conditional on age are given by*

$$\mathbb{E}[\tilde{n}_{i,t}|a_i = a] = \tilde{h}_0 e^{\left( r + \frac{(p^{as})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A \right) a} \frac{e^{-\bar{r}a} - e^{-\bar{r}T}}{1 - e^{-\bar{r}T}} - \tilde{h}_a \quad (27)$$

$$\mathbb{V}[\tilde{n}_{i,t}|a_i = a] = \left[ e^{\left( \frac{p^{id}}{\gamma} \right)^2 a} - 1 \right] \left[ \tilde{h}_0 e^{\left( r + \frac{(p^{as})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A \right) a} \frac{e^{-\bar{r}a} - e^{-\bar{r}T}}{1 - e^{-\bar{r}T}} \right]^2 \quad (28)$$

iii. **Inverted-U shape of inequality over the life cycle:** *there exists  $0 < \hat{a} < T$  such that  $\mathbb{V}[\tilde{n}_{i,t}|a_i = a]$  is increasing in  $a$  for  $a < \hat{a}$  and decreasing for  $a > \hat{a}$ .*

Proposition 3 gives a complete characterization of the distribution of wealth conditional on age. Wealth has a *shifted log-normal* distribution, with an age dependent shifter  $-\tilde{h}_a$ . Since entrepreneurs

are allowed to borrow, financial wealth clearly cannot be log-normally distributed, as  $\tilde{n}_{i,t}$  can take on negative values. However, financial wealth cannot go below the natural borrowing limit  $-\tilde{n}_{i,t}$ , so total wealth will assume only positive values. Total wealth follows a log-normal distribution with a growth rate which varies with age.

Expression (27) is essentially equation (118) rearranged and specialized to the case  $\psi = 1$ .<sup>13</sup> This can be seen by noting that the MPCs cumulated up to age  $a$  is given by  $\int_0^a \frac{\bar{r}}{1-e^{-\bar{r}(T-\tilde{a})}} d\tilde{a} = -\log \frac{e^{-\bar{r}a} - e^{-\bar{r}T}}{1-e^{-\bar{r}T}}$ . As we have seen, average wealth tends to increase with age at the beginning of life and it goes down by the end of the life cycle, as the result of the increase in MPC balancing out the effect of wealth being accumulated over time.

Expression (28) shows how the variance of wealth evolves over the life cycle. In the case with no bequest, the variance is zero at ages  $a = 0$  and  $a = T$ . Since entrepreneurs leave no bequests, everyone starts with zero financial wealth. By the end of life the MPC becomes arbitrarily large, so the flow of consumption exhausts the whole stock of wealth. The dispersion of wealth increases at the beginning of the life cycle. This is the result of the fact that some entrepreneurs will be lucky and receive a series of positive shocks, while others will suffer a sequence of negative shocks. This force is magnified with the exposure to idiosyncratic risk  $p^{id}/\gamma$ , but also with the magnitude of portfolio returns net of the growth rate, i.e., a version of " $r - g$ ". The increase in MPC provides a countervailing force, as the impact of the proportional shocks is reduced as the level of wealth is brought down at the end of the life cycle.

Figure 5 shows the evolution of within group inequality over the life cycle in the Thai data. It shows how the standard deviation of  $\tilde{n}_{i,t}$ , financial wealth normalized by the average wealth of entrepreneurs, increases sharply from ages 40 to age 55 and then declines until the end of the cycle.

## 4 Equilibrium

In order to determine the equilibrium prices,  $(r, p^{as}, q, w)$  and aggregate capital-labor ratio  $k$ , we need to close the model by specifying the market clearing conditions and the agents who will provide insurance and funds to entrepreneurs, which we will refer to as *financiers*.

### 4.1 Equilibrium definition

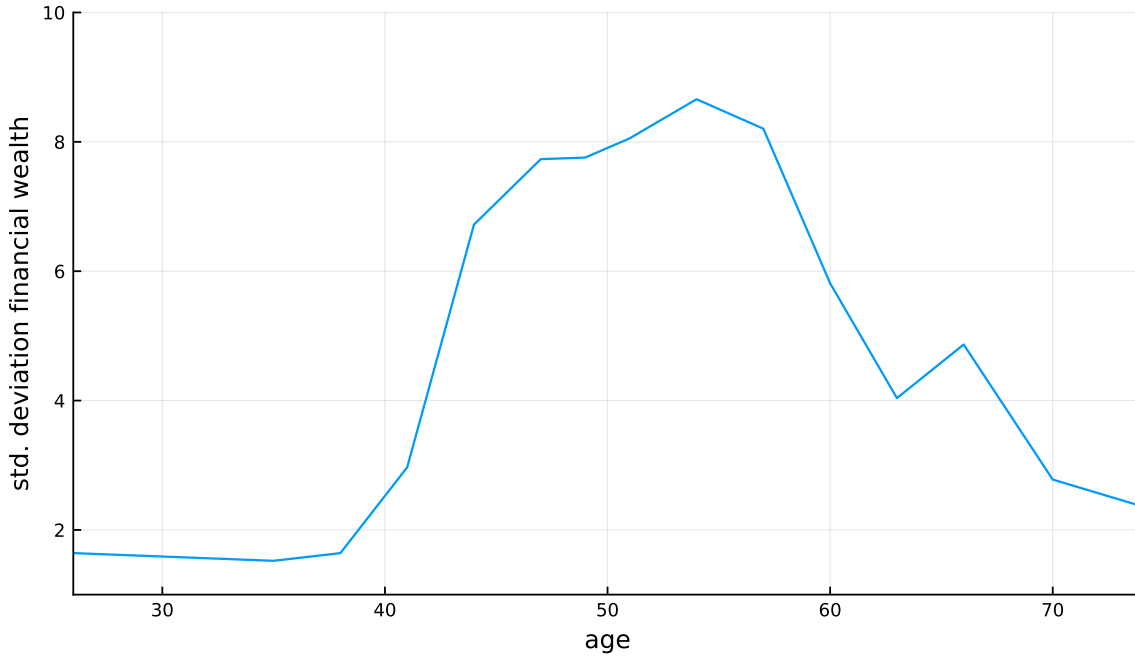
#### Financiers

Financiers choose a path of consumption,  $c_f$ , how much of aggregate insurance to provide,  $\theta_{f,t}^{as}$ , given labor income  $w_t \bar{l}_{f,t}$  and initial value of net assets  $n_{f,0}$ . Labor endowment of financiers  $\bar{l}_{f,t}$  grows at rate  $g$ . Financiers maximize utility subject to the law of motion of wealth

$$dn_{f,t} = \left[ r_t n_{f,t} + p^{as} \theta_{f,t}^{as} + w_t \bar{l}_f - c_{f,t} \right] dt + \theta_{f,t}^{as} dZ_t \quad (29)$$

<sup>13</sup>One difference is the fact that  $n_a = f(a)\mathbb{E}[\tilde{n}_{i,t}|a_i = a]$ , i.e., it takes into account not only the average wealth of entrepreneurs of age  $a$ , but also the mass of such agents.

**Figure 5:** Standard deviation of financial wealth within age groups



a natural borrowing constraint  $n_{f,t} \geq -h_{f,t}$  and given initial net worth  $n_{f,0} > -h_{f,0}$ .

In order to keep the problem of financiers as simple as possible, we will assume they are infinitely lived and have unit elasticity of intertemporal substitution, so the consumption function is given by the standard formula  $c_{f,t} = \rho_f(n_{f,t} + h_{f,t})$ , where  $\rho_f > 0$  is the discount rate of financiers and  $h_{f,t}$  is the human wealth. The risk aversion parameter is  $\gamma > 0$ . The expression for the optimal provision of insurance is analagous to the demand derived for entrepreneurs, except for sign changes:

$$\frac{\theta_{f,t}^{ag}}{n_{f,t}} = \frac{1 + \frac{h_{f,t}}{n_{f,t}}}{\gamma} p^{ag} - \frac{h_{f,t}}{n_{f,t}} \sigma_A \quad (30)$$

where  $h_{f,t}$  denotes the human wealth of financiers.

### Competitive Equilibrium

A *competitive equilibrium* is a sequence of allocations  $(c_i, k_i, l_i, t_i, \theta_i^{ag}, \theta_i^{id}, c_f, \theta_f^{ag})$  and prices  $(r, p^{ag}, q, w)$  such that

- (a)  $(c_i, k_i, l_i, t_i, \theta_i^{ag}, \theta_i^{id})$  solves the entrepreneur's problem (9), given  $(r, p^{ag}, q, w)$
- (b)  $(c_f, \theta_f^{ag})$  solves the financier's problem, given  $(r, p^{ag})$
- (c) Market clearing:

i. Goods market

$$\int_{t-T}^t [c_{s,t} + \Phi(l_{s,t})A_t k_{s,t}] ds + c_{f,t} = \int_{t-T}^t y_{s,t} ds$$

ii. Financial markets

$$\int_{t-T}^t (n_{s,t} - q_t k_{s,t}) ds + n_{f,t} = 0; \quad \int_{t-T}^t \theta_{s,t}^{ag} ds = \theta_{f,t}^{ag}$$

iii. Factor markets

$$\int_{t-T}^t l_{s,t} dz = \int_{t-T}^t \bar{l}_{s,t} ds + \bar{l}_f; \quad \int_{t-T}^t k_{s,t} ds = k_t$$

The first market clearing condition corresponds to the goods market, implying that the value of consumption plus investment made by entrepreneurs and the consumption of financiers must equal total output produced in the economy. The second set of conditions gives the market clearing for the riskless asset and for aggregate insurance. Since bonds are in zero net supply, the net worth of entrepreneurs and financiers must add to the asset in positive net supply, the capital stock. Finally, we have the market clearing condition for capital and labor.

## 4.2 Equilibrium characterization

Let's now present the equilibrium characterization. First, we will discuss the determination of the price of aggregate insurance, interest rate, and wages, the last one as function of the level of capital. Then, we will discuss how the capital stock and the idiosyncratic risk premium are simultaneously determined. Proofs and detailed calculations are provided in the appendix.

### Price of aggregate insurance

The net demand for insurance in the economy is given by

$$\int_{t-T}^t \theta_{s,t}^{ag} ds - \theta_{f,t}^{ag} = \left( \int_{t-T}^t (q_t k_{s,t} + h_{s,t}) ds + h_{f,t} \right) \sigma_A - \left( \int_{t-T}^t (n_{s,t} + h_{s,t}) ds + n_{f,t} + h_{f,t} \right) \frac{p^{ag}}{\gamma}$$

From the market clearing for bonds, the net demand for insurance will be equal to zero if

$$p^{ag} = \gamma \sigma_A \quad (31)$$

### Wages and the relative price of capital

In a stationary equilibrium, the capital-labor ratio is constant, so capital grows at rate  $g$ . This requires an investment rate equal to  $g + \delta$ . The price of capital must then satisfy

$$l_{i,t} = \frac{q - \bar{\Phi}_0}{\bar{\Phi}_1} \Rightarrow q = \bar{\Phi}_0 + \bar{\Phi}_1(g + \delta) \quad (32)$$

Given the aggregate capital-labor ratio, wages are determined by the labor demand condition

$$w = (1 - \alpha)k^\alpha \quad (33)$$

### Interest rate

In a stationary equilibrium, the growth rate of the financier's total wealth must be equal to  $g + \mu_A$ , the growth rate of the economy

$$r + \gamma\sigma_A^2 - \rho_f = g + \mu_A$$

using the fact that  $\theta_{f,t}^{ag} = \sigma_A$  and  $p^{ag} = \gamma\sigma_A$ .

Rearranging the expression above, we can solve for the interest rate

$$r = \rho_f + (g + \mu_A) - \gamma\sigma_A^2 \quad (34)$$

### Aggregate capital stock and the price of idiosyncratic risk

From equation (18), we obtain the expression

$$r + p^{ag}\sigma_A + p^{id}\phi\sigma_{id} = \frac{\alpha k^{\alpha-1} - \Phi(g + \delta)}{q} + g + \mu_A \quad (35)$$

The left-hand side captures the required rate of return of investing in the business, which includes a premium for holding aggregate and idiosyncratic risk. The right-hand side gives the actual expected return of investing in the business, a function of the marginal product of capital net of adjustment costs. Notice expression (35) generalizes the standard relation between marginal product of capital and interest rates to an environment with growth, risk, and adjustment costs. In the absence of risk,  $\sigma_A = \sigma_{id} = 0$ , growth,  $g = \mu_A = 0$ , and adjustment costs,  $\bar{\Phi}_0 = 1, \bar{\Phi}_1 = 0$ , the expression above boils down to  $r = \alpha k^{\alpha-1} - \delta$ . The usual result that higher interest rates leads to a reduction in the capital-labor ratio translates here to a negative relationship between returns, inclusive of risk premium, and the capital-labor ratio.

The prices  $r$ ,  $p^{ag}$ , and  $q$  are entirely determined by a small set of parameters, independent of the value of  $k$ . Hence, expression (35) essentially gives an inverse relation between the price of idiosyncratic risk and the stock of capital in the economy.

### The determination of the idiosyncratic risk premium

In order to determine the idiosyncratic risk premium in this economy, we need another equation relating  $k$  and  $p^{id}$ . First, aggregating the demand for capital (19) across all entrepreneurs, we obtain

$$p^{id} = \underbrace{\gamma}_{\text{risk aversion}} \underbrace{\phi\sigma_{id}}_{\text{effective risk}} \underbrace{\frac{q_t k}{n_{e,t} + h_{e,t}}}_{\text{leverage factor}} \quad (36)$$

where  $n_{e,t}$  and  $h_{e,t}$  denote the average financial and human wealth of entrepreneurs.

In the same way the price of aggregate risk depends on the product of risk aversion and risk,  $\gamma\sigma_A$ , the price of idiosyncratic risk also depends on the product of risk aversion  $\gamma$  and effective risk (risk net of insurance)  $\phi\sigma_{id}$ . However, the price of idiosyncratic risk depends on an additional term: the *leverage factor*, i.e., the ratio of physical assets to total wealth of entrepreneurs. This term captures the fact that entrepreneurs will require a larger idiosyncratic risk premium the higher the fraction of their wealth is invested in the business. This term is larger than one if entrepreneurs collectively borrow from financiers to invest in the business and it is less than one if entrepreneurs are instead net lenders. The leverage factor can be decomposed into

$$\frac{q_t k}{n_{e,t} + h_{e,t}} = \frac{qk}{qk + h} \frac{1}{x_e}$$

where  $h$  is total human wealth in the economy, normalized by  $A_t e^{gt}$ , and  $x_e$  is the share of the total wealth in the economy held by entrepreneurs. Hence, the leverage factor depends on the ratio of the fraction of total wealth in physical form to the fraction of wealth in the hands of entrepreneurs.

It can be shown that human wealth is a multiple of wages, so  $h$  is a function of  $k$

$$h = \chi_h (1 - \alpha) k^\alpha$$

where  $\chi_h$  is a function of the parameters given in the appendix.

The ratio of physical to total wealth,  $qk/(qk + h)$ , is entirely determined by the capital stock  $k$ . It remains to solve for  $x_e$ . Using the fact the wealth of entrepreneurs must grow at the same rate as the economy in a stationary equilibrium, we obtain the condition

$$r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \int_0^T \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}} \omega_a da + \frac{\hat{h}_0}{x_e} = g + \mu_A \quad (37)$$

where  $\hat{h}_0$  is the ratio of the human wealth of newborn agents to total wealth in the economy.

The growth rate of wealth for entrepreneurs depends on three terms. The return on the portfolio,  $r + (p^{ag})^2/\gamma + (p^{id})^2/\gamma$ , the average MPC weighted by total wealth  $\int_0^T \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}} \omega_a da$ , and a term capturing a composition effect  $\hat{h}_0/x_e$ . As old entrepreneurs are replaced by new ones, the amount of wealth increases by the human wealth of newborn agents. After some tedious derivation, we can use equation (37) to solve  $x_e$  as a function of  $p^{id}$ . Given a value for  $x_e$ , we can use (35) and (36) to solve for  $p^{id}$  and  $k$ .

### 4.3 Calibration

We adopt the following calibration. The capital share is set to  $\alpha = 0.3$  and the average growth rate of productivity is set to  $\mu = 0.003$ , following the evidence provided by Jeong and Townsend (2007) for Thailand. Aggregate and idiosyncratic volatility are set to  $\sigma = 0.053$  and  $\sigma_{id} = 0.196$ , using the estimates in Samphantharak and Townsend (2018). The adjustment costs are chosen to match an investment rate of 20% and relative price of capital of one. The depreciation rate is set to  $\delta = 0.10$ . The discount rate of financiers is chosen to match a risk-free interest of  $r = 0.03$ . The discount rate of en-

trepreneurs and the bequest motive parameters are chosen to match the consumption-wealth ratio at the beginning and end of life. The life horizon is set to  $T = 55$ , so it covers the life span from 25 to 80 years old, and the population growth is set to  $g = 0.003\%$ , the most recent value for population growth in Thailand. The moral hazard parameter  $\phi$  can be identified from information on the idiosyncratic risk premium. We set it to  $\phi = 0.75$  to capture the level of idiosyncratic risk premium.

#### 4.4 The price of aggregate and idiosyncratic risk

The model is able to replicate the risk premia and volatility on aggregate and idiosyncratic risk. Table 1 gives the risk and return decomposed by the source of risk. A striking fact is that despite idiosyncratic volatility being three times larger than aggregate volatility, accounting for more than 90% of total variance, the idiosyncratic risk premium is only slightly bigger than the aggregate risk premium. This will lead to a Sharpe ratio three times larger for aggregate risk. The pricing equations (31) and (36) will help to shed light on this pattern.

**Table 1:** Aggregate and idiosyncratic components of risk and return

	Risk premium	% of returns	Volatility	% of variance	Sharpe ratio
Total returns	4.1%	100%	20.3%	100%	0.20
Aggregate component	1.9%	46.3%	5.3%	6.8%	0.35
Idiosyncratic component	2.2%	53.7%	19.6%	93.2%	0.11

The Sharpe ratio of aggregate risk corresponds to  $p^{as}$  and it is given by  $\gamma\sigma_A$ . The Sharpe ratio of idiosyncratic risk is given by  $p^{id}\phi$ , as the volatility reported here does not take into account any insurance taken by the entrepreneur. If we were to naively price the idiosyncratic risk by analogy with the aggregate risk, the Sharpe ratio would be  $\gamma\sigma_{id}$ , i.e., more than three times *larger* than the one for aggregate risk. Two factors explain why the price of idiosyncratic risk is actually three times *smaller* than the one for aggregate risk: idiosyncratic insurance and the leverage factor. The pricing equation for idiosyncratic risk shows the role of these two components:

$$p^{id}\phi = \gamma\sigma_{id}\phi^2 \frac{qk_t}{n_{e,t}(1 + h_{e,t}/n_{e,t})}$$

The parameter of the skin-in-the-game constraint is given by  $\phi = 0.75$ . This term by itself will reduce the price of idiosyncratic risk by 45%. However, the bulk of the adjustment comes from the leverage factor, as  $qk/(n_e + h_e) \approx 0.17$ . This is the result of a share invested in the business of around 25% and human-financial wealth ratio of around 0.5 on average. Intuitively, the reason for a much smaller price of idiosyncratic risk is that the entrepreneurs are proportionally less exposed to idiosyncratic risk, either because of insurance mechanisms, or because only a fraction of total wealth is exposed to this risk. Notice the importance of explicitly introducing human wealth and heterogeneous agents into the model for the pricing of idiosyncratic risk. In an environment without human wealth and with a representative agent, the leverage factor would be necessarily equal to one, thus ignoring an important determinant of the idiosyncratic risk premium.



## 5 Counterfactuals

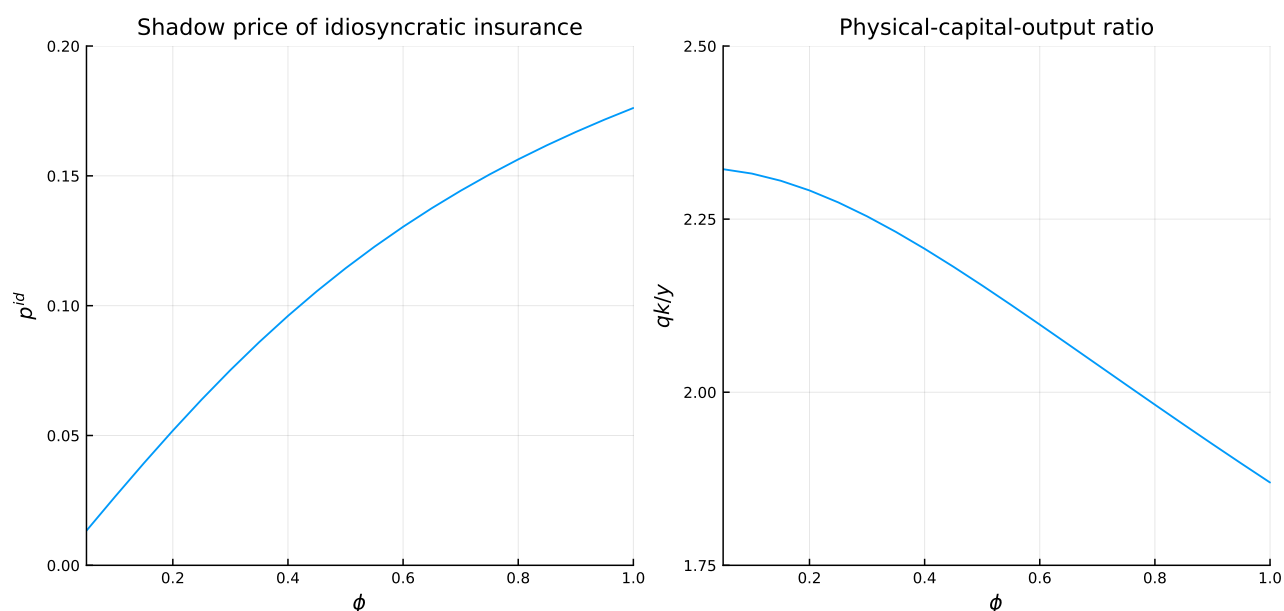
Having disciplined the model parameters using the micro data on entrepreneurs, we can now use the model to evaluate the consequences of changes in the environment. We will focus on two main counterfactuals: the process of financial development, captured by a reduction in the skin-in-the-game parameter  $\phi$ , and the consequences of demographic transitions, captured by changes in the population growth rate or the life expectancy parameter  $T$ .

### 5.1 Financial development

The parameter  $\phi$  measures the strength of contractual frictions in the economy. High values of  $\phi$  are meant to capture situations where the access to insurance arrangements, formal or informal, is rather limited. In this case, entrepreneurs are forced to hold most of the idiosyncratic risk of their businesses, potentially limiting their choice of scale. As the institutional arrangements improve, in particular mechanisms to monitor the entrepreneur's activities, such frictions are expected to be reduced and entrepreneurs would hold a smaller fraction of the risk. Hence, we see the process of financial development as leading to a reduction in the skin-in-the-game parameter  $\phi$ .

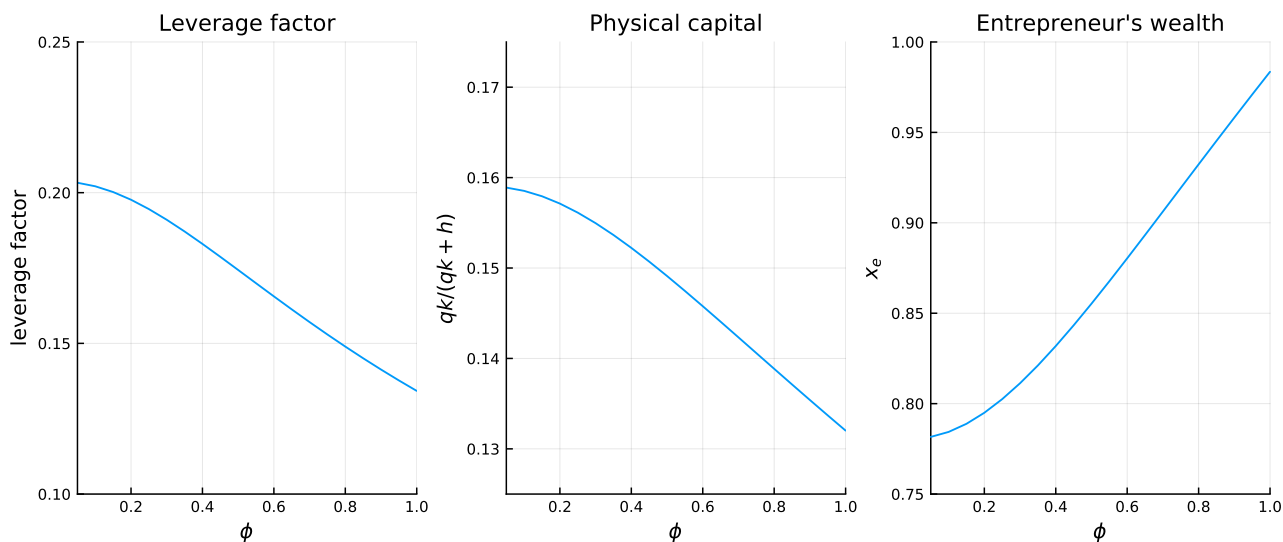
Figure 6 shows how the shadow price of idiosyncratic insurance and the capital-output ratio vary with the parameter  $\phi$ . A reduction in  $\phi$  will lead to a reduction in  $p^{id}$  and in the idiosyncratic risk premium. As the required return of investing in the business falls, the capital stock in the economy increases, so the marginal product of capital will match the required return, as can be seen in (35). Hence, financial development and economic development go together in this economy. Consider reducing the skin-in-the-game parameter from  $\phi = 0.75$  to  $\phi = 0.5$ , so the idiosyncratic risk premium falls by a little more than 1%. This implies an increase of about 10% in the capital stock.

**Figure 6:** Price of idiosyncratic insurance and the stock of capital



Notice that the price of idiosyncratic risk does not fall linearly with  $\phi$ . From expression (36), the price of idiosyncratic risk depends on  $\phi$  but also on the leverage factor, or the total exposure of entrepreneurs to the business. Figure 7 shows that the leverage factor increases in this economy as  $\phi$  decreases. The leverage factor increases in part because physical wealth becomes relatively more important, as the capital-output ratio increases, but human wealth-output ratio stays the same. Moreover, the share of wealth of entrepreneurs fall in response to a drop in  $\phi$ , amplifying the effect of the increase in the importance of capital.

**Figure 7:** The leverage factor and its components



Financial development also has strong implication for inequality. First, entrepreneurs will hold a smaller fraction of the wealth. This is the result of the reduction in the risk premium, as entrepreneurs will accumulate wealth less fast compared to financiers. Changes in  $\phi$  will also affect how wealth is distributed across entrepreneurs. Figure 8 shows the impact of financial development on between and within group inequality.<sup>14</sup> The impact on the between group inequality is small, on average entrepreneurs accumulate roughly the same amount of wealth over the life cycle in relative terms. But the within-group inequality changes dramatically. There is a substantial decline in inequality within a given age group. This is a consequence of the reduction of the shadow price of idiosyncratic insurance, as the exposure to idiosyncratic risk is given by  $p^{id}/\gamma$ .

## 5.2 Demographic transitions

Consider now the effect of demographic transitions. Changes in the population growth rate or in life expectancy will affect the shape of the age distribution in the population. Given the substantial heterogeneity in the risk-taking and savings behavior over the life cycle, it is expected that such changes will have aggregate and distributive impacts.

<sup>14</sup>The within group inequality measure uses the formula for the case  $\psi = 1$ .

**Figure 8:** Financial development and inequality

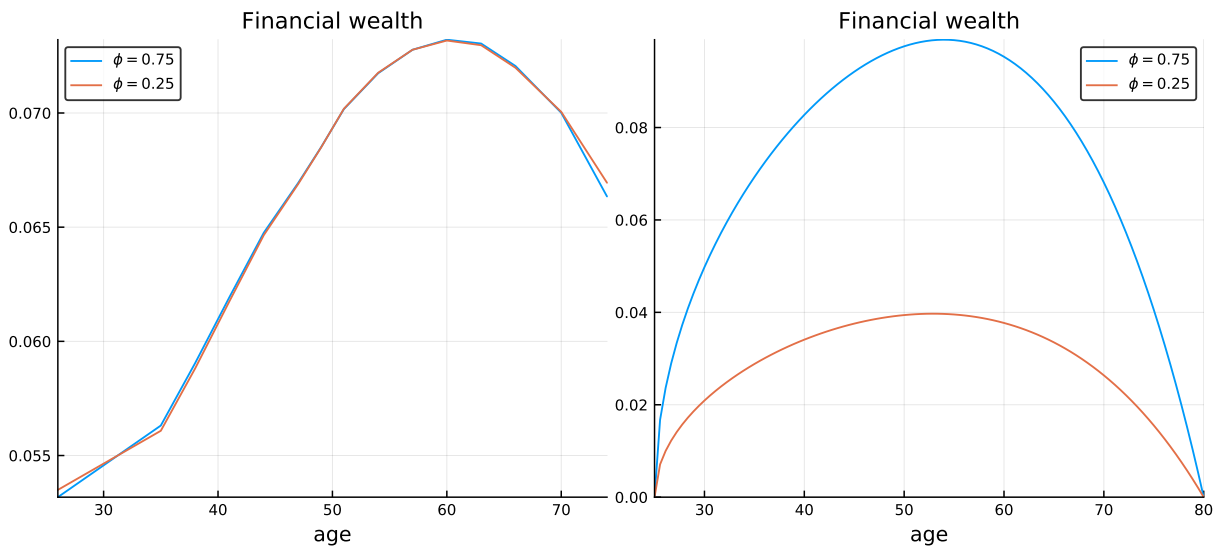
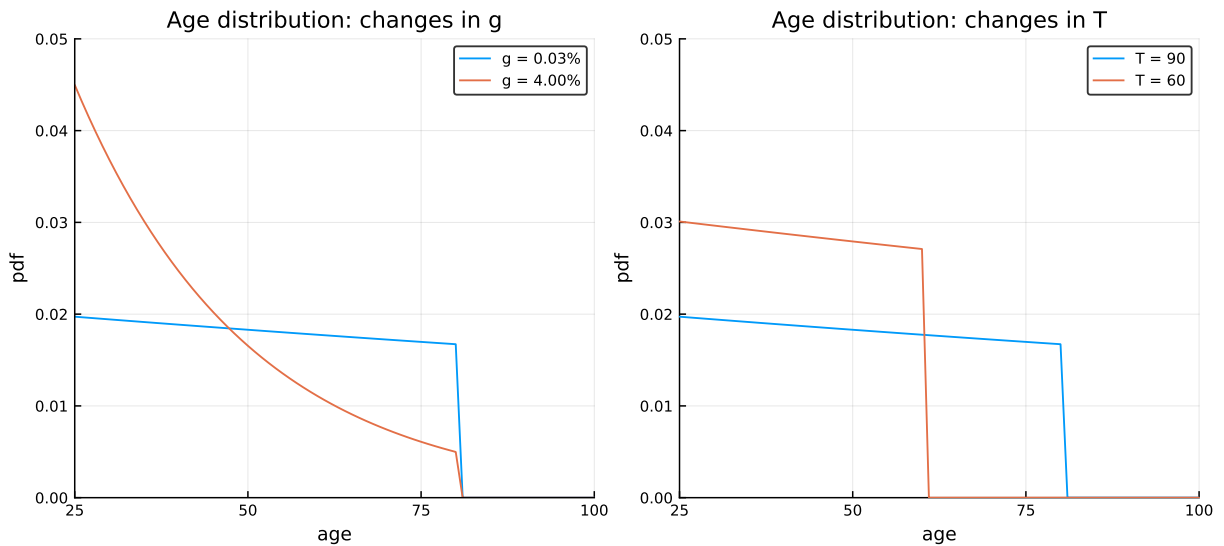


Figure 9 shows how the age distribution responds to changes in  $g$  or  $T$ . A reduction in the population growth rate or an increase in life expectancy will have the effect of shifting mass from young to old people. Moreover, a reduction in population growth rate will reduce the growth rate of the economy.

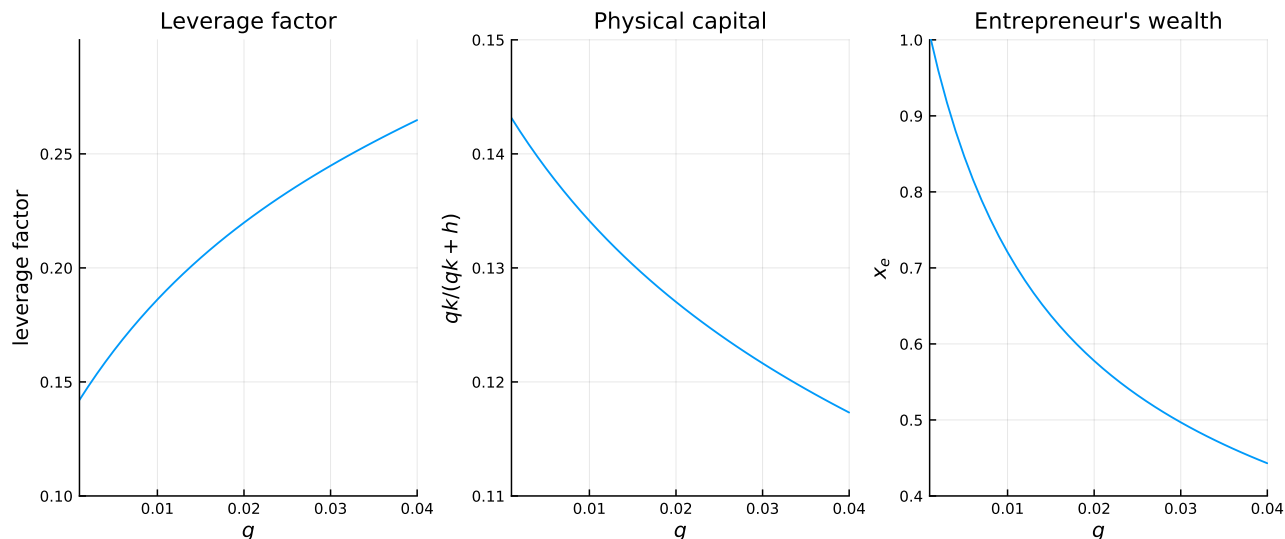
**Figure 9:** Demographic changes: population growth and life expectancy



An implication of the reduction in the growth rate is that the real interest rate will fall. This will tend to reduce the consumption-wealth ratio of entrepreneurs, causing them to accumulate more wealth in the long-run, as shown in figure 10. As the share of wealth of entrepreneurs increase, this will bring down the leverage factor, reducing the price of idiosyncratic risk, and stimulating investment. This illustrates a stark difference between the pricing of aggregate and idiosyncratic risk. While the price

of aggregate risk depends only on risk and risk aversion, the price of idiosyncratic risk depends on broader economic conditions, in particular, the demographic structure.

**Figure 10:** The leverage factor and its components



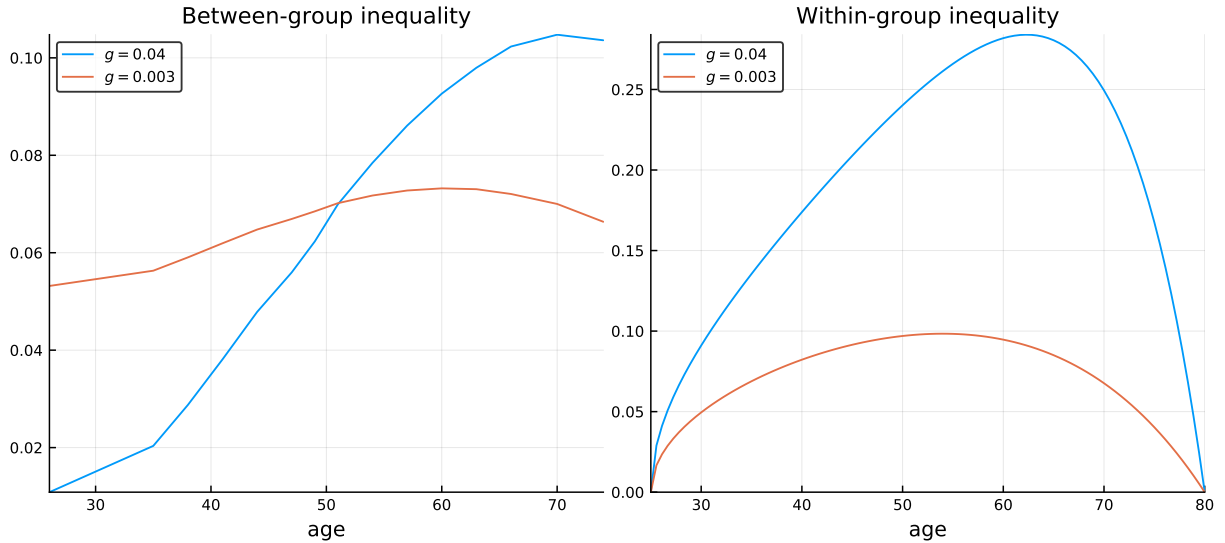
The interaction between savings and risk-taking behavior will be important to understand the consequences of demographic changes in inequality. In an economy with fast population growth, the bequests get diluted among a larger group of heirs, what explains the steeper wealth accumulation pattern in this case. Within group inequality falls sharply in an economy with low population growth. The reason is again the reduction in the price of idiosyncratic insurance, which will affect the risk exposure of entrepreneurs.

Notice that a partial equilibrium view of the problem would be very misleading in this case. If one considers “ $r - g$ ”, or  $r + p^{ag}\sigma_A + p^{id}\phi\sigma_{id} - (g + \mu)$  more generally, a reduction in  $g$  would be thought to increase this difference when we take returns as given. The increase in  $r - g$  would then potentially increase inequality. The problem is that returns respond to changes in demographics. First, the interest rate will fall with  $g$ . In the case of unit intertemporal elasticity of substitution we assume here, the interest rate will fall by the exact same amount as  $g$ . Hence,  $r + p^{ag}\sigma_A + p^{id}\phi\sigma_{id} - (g + \mu)$  will move only because of changes in the price of idiosyncratic risk. As we have seen,  $p^{id}$  will fall in response to a reduction in population growth, so the generalized “ $r - g$ ” relation actually falls with  $g$ , instead of increasing. Understanding the response of the price of idiosyncratic risk to different shocks is then crucial to understand how shocks affect aggregate and distributive outcomes.

## 6 Conclusion

In this paper, we provide a framework to analyze the risk-taking behavior of entrepreneurs over the life cycle. We show that the risk-taking decision depends on two main factors: the human-financial wealth ratio and the shadow price of idiosyncratic insurance. These two components can be recovered in the

**Figure 11: Demographic changes and inequality**



data by using information on returns and labor income. We document that the risk-taking and savings decisions vary substantially over the life cycle. Risk-taking falls with age while the consumption-wealth ratio is U-shaped. These life cycle patterns can be explained by a decline in the human-financial wealth ratio and the increase in MPC as entrepreneurs approach the end of life.

We consider also the implications of entrepreneur's risk taking for wealth inequality. The effects on inequality can be divided into between-group inequality, where we consider differences in average wealth across different age groups, and within-group inequality, where we consider differences in wealth within a given age group. Between-group inequality initially increases with age and then declines. Within-group inequality has also a U-shaped life cycle pattern. Both predictions are supported by the data.

Finally, we consider the equilibrium determination and counterfactuals. We show that the process of financial development usually comes associated with economic development, as the capital stock in the economy increases in response to improvements in the availability of idiosyncratic insurance to entrepreneurs. Moreover, financial development causes within-group inequality to fall, as the shadow price of idiosyncratic insurance falls. Similarly, a demographic transition that reduces the population growth will lead to increase in output and decline inequality.

In this paper, we consider the long-run consequences of such changes in the economic environment. Computing the transitional dynamics and considering welfare implications are natural next steps.

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## A Derivation of Optimal Contract

### A.1 From discrete to continuous time

*Prices:* The relative price of capital  $q_t$  and the stochastic discount factor can be written as:

$$\frac{q_{t+\Delta}}{q_t} = 1 + \mu_{q,t}\Delta + \sigma_{q,t}z_{t+\Delta}\sqrt{\Delta}; \quad \frac{\pi_{t+\Delta}}{\pi_t} = 1 - r_t\Delta - \eta_t z_{t+\Delta}\sqrt{\Delta};$$

or in levels

$$q_{t+\Delta} = q_0 + \sum_{j=0}^t \mu_{q,j}q_j\Delta + \sum_{j=0}^t \sigma_{q,j}q_j z_{t+j}\sqrt{\Delta}; \quad \pi_{t+\Delta} = \pi_0 - \sum_{j=0}^t r_j\pi_j\Delta - \sum_{j=0}^t \eta_j q_j z_{t+j}\sqrt{\Delta};$$

Taking the limit as  $\Delta$  goes to zero:

$$q_t = q_0 + \int_0^t \mu_{q,u}q_u du + \int_0^t \sigma_{q,u}q_u dZ_u; \quad \pi_t = \pi_0 - \int_0^t r_u\pi_u du - \int_0^t \eta_u\pi_u dZ_u;$$

Using the usual more compact notation, we obtain

$$dq_t = \mu_{q,t}dt + \sigma_{q,t}dZ_t; \quad d\pi_t = -r_tdt - \eta_t dZ_t \quad (38)$$

where  $Z = \{Z_t : 0 \leq t \leq T\}$  is a Brownian motion.

In order to see the intuition for the approximation above, notice we can write

$$\lim_{\Delta \rightarrow 0} \sum_{j=0}^t z_j \sqrt{\Delta} = \sqrt{t} \lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \sum_{j=0}^t z_j \equiv Z_t \sim \mathcal{N}(0, t) \quad (39)$$

where  $\Delta = t/N$ .

*Returns:* Define the cumulative log return:

$$\tilde{R}_{a,t+\Delta} \equiv \log \prod_{j=0}^t R_{i,j+\Delta} \quad (40)$$

Changes in cumulative return are given by<sup>15</sup>

$$\tilde{R}_{a,t+\Delta} - \tilde{R}_{a,t} = \left[ \frac{ak_{a,t}^\alpha l_{a,t}^{1-\alpha} - \iota(g_{a,t})k_{a,t}}{q_{a,t}k_{a,t}} + \mu_{q,t} + \sigma\sigma_{q,t} + g_{a,t} \right] \Delta + (\sigma + \sigma_{q,t})z_{t+\Delta}\sqrt{\Delta} + v\omega_{a,t+\Delta}\sqrt{\Delta} + o(\Delta)$$

where  $o(\Delta)$  collects terms such that satisfy  $\lim_{\Delta \rightarrow 0} \frac{o(\Delta)}{\Delta} = 0$ .

<sup>15</sup>The expression for the return already imposes  $s_{a,t+\Delta} = 0$ .



Taking the limit as  $\Delta \rightarrow 0$ :

$$d\tilde{R}_{a,t} = \left[ \underbrace{\frac{ak_{a,t}^\alpha l_{a,t}^{1-\alpha} - \iota(g_{a,t})k_{a,t}}{q_t k_{a,t}} + \mu_{q,t} + \sigma\sigma_{q,t} + g_{a,t}}_{\mu_{a,t}} \right] dt + (\sigma + \sigma_{q,t})dZ_t + \nu dW_{a,t} \quad (41)$$

I will abuse notation and write  $dR_{a,t}$  instead of  $d\tilde{R}_{a,t}$  and similarly for the market return.

*State contingent liability:* The participation constraint can be written as

$$0 = \mathbb{E} \left[ \frac{\pi_{t+\Delta}}{\pi_t} \left( \bar{m}_t \Delta + \beta_{t,z} z_{t+\Delta} \sqrt{\Delta} + (1 - \phi) q_{t+\Delta} k_t w_{t+\Delta} \sqrt{\Delta} \right) \right]$$

Using the expression for the stochastic discount factor and ignoring higher order terms in  $\Delta$ :

$$\bar{m} = \beta_{t,z} \eta_t$$

For the continuous time analysis it is more convenient to define the cumulative payment to the principal  $M_t = \sum_{j=0}^t m_j$  such that:

$$dM_t = \bar{m} dt + \beta_{t,z} dZ_t + (1 - \phi) q_t k_t dW_t$$

*Net worth evolution:* From the flow budget constraint and plugging in the expression for  $m_{t+\Delta}$ , we obtain

$$\begin{aligned} \frac{n_{t+\Delta} - n_t}{n_t} &= \left[ r_t - \frac{c_t}{n_t} - \frac{\beta_{t,z}}{n_t} \eta_t + \frac{q_t k_t}{n_t} (\mu_t - r_t) \right] \Delta - \frac{\beta_{t,z}}{n_t} z_{t+\Delta} \sqrt{\Delta} \\ &\quad + \frac{q_{t+\Delta} k_t}{n_t} \frac{q_{t+\Delta}}{q_t} \left[ 1 + \sigma z_{t+\Delta} \sqrt{\Delta} + \phi v w_{t+\Delta} \sqrt{\Delta} \right] \end{aligned}$$

Taking the limit as  $\Delta$  goes to zero:

$$\frac{dn_t}{n_t} = \left[ r_t - \frac{c_t}{n_t} + \sigma_{agg,t} \eta_t + \phi v \frac{q_t k_t}{n_t} \eta_{id,t} \right] dt + \sigma_{agg,t} dZ_t + \phi v \frac{q_t k_t}{n_t} dW_t$$

where

$$\sigma_{agg,t} \equiv \frac{q_t k_t}{n_t} (\sigma + \sigma_{q,t}) - \frac{\beta_{t,z}}{n_t}; \quad \eta_{id,t} \equiv \frac{\mu_t - r_t - (\sigma + \sigma_{q,t}) \eta_t}{\phi v}$$

*Optimal contract in continuous time:*

$$\rho V_t(n) dt = \max_{c,l,g,k,\sigma_{agg}} u(c,l) dt + \mathbb{E} [dV_t(n)]$$

subject to

$$\frac{dn_t}{n_t} = \left[ r_t + \sigma_{agg,t} \eta_t + \phi v \frac{q_t k_t}{n_t} \eta_{id,t} - \frac{c_t}{n_t} \right] dt + \sigma_{agg,t} dZ_t + \phi v \frac{q_t k_t}{n_t} dW_t$$

$$\eta_{id,t} = \frac{\mu_t - r_t - (\sigma + \sigma_{q,t})\eta_t}{\phi\nu}; \quad \mu_t = \frac{ak_{a,t}^\alpha l_{a,t}^{1-\alpha} - \iota(g_{a,t})k_{a,t}}{q_t k_{a,t}} + \mu_{q,t} + \sigma\sigma_{q,t} + g_t; \quad n_t \geq 0$$

and the law of motion of  $(r_t, \eta_t, q_t)$ .

## B Proofs

### B.1 Proof of lemma 1 and proposition 1

*Proof.* We will start by showing part b of lemma 1, i.e., we are going to solve for  $h_{i,t}$ . Then, we will proceed to solve for the entrepreneur's value function which will give us the rest of lemma 1 and proposition 1.

**Human wealth:** Define the stochastic discount factor for this economy as the process  $\pi_t$  satisfying the law of motion

$$\frac{d\pi_t}{\pi_t} = -r_t dt - p_t^{ag} dZ_t \quad (42)$$

The discounted value of labor income for an entrepreneur since he was born is given by:

$$G_{i,t} = \int_s^t \pi_z w_z \bar{l}_{i,z} dz + \underbrace{\mathbb{E}_t \left[ \int_t^{s_i+T} \pi_z w_z \bar{l}_{i,z} dz \right]}_{=\pi_t h_{i,t}} \quad (43)$$

for  $s \leq t \leq s + T$ , and zero otherwise.

Notice the term  $G_{i,t}$  is a martingale, since

$$\mathbb{E}_t[G_{i,t'}] = \int_{s_i}^{t'} \pi_z w_z \bar{l}_{i,z} dz + \mathbb{E}_t \left[ \int_t^{t'} \pi_z w_z \bar{l}_{i,z} dz \right] + \mathbb{E}_t \left[ \mathbb{E}_{t'} \left[ \int_{t'}^{s_i+T} \pi_z w_z \bar{l}_{i,z} dz \right] \right] = G_{i,t}$$

from the law of iterated expectations.

From the martingale representation theorem, there exists a process  $\sigma_{i,t}^h$  such that

$$\pi_t w_t \bar{l}_{s,t} dt + d(\pi_t h_{i,t}) = \pi_t h_{i,t} (\sigma_{i,t}^h - p_t^{ag}) dZ_t \quad (44)$$

Applying Ito's lemma, we obtain

$$dh_{i,t} = \left[ r_t h_{i,t} + \sigma_{i,t}^h h_{i,t} p_t^{AG} - w_t \bar{l}_{i,t} \right] dt + \sigma_{i,t}^h h_{i,t} dZ_t \quad (45)$$

Let's now show that in a stationary equilibrium  $\sigma_{i,t}^h = \sigma_A$ . The stochastic discount factor is given by

$$\frac{d\pi_t}{\pi_t} = -r dt - p^{ag} dZ_t \Rightarrow \pi_z = e^{-\left(r + \frac{(p^{ag})^2}{2}\right)(z-t) - p^{ag}(Z_z - Z_t)} \pi_t \quad (46)$$

The wage is given by  $w_z = e^{\left(\mu_A - \frac{\sigma_A^2}{2}\right)(z-t) + \sigma_A(Z_z - Z_t)} w_t$ . Human wealth is then given by

$$\begin{aligned}
h_{i,t} &= \mathbb{E}_t \left[ \int_t^{s_i+T} \frac{\pi_z}{\pi_t} w_z \bar{l}_{i,z} dz \right] \\
&= \mathbb{E}_t \left[ \int_t^{s_i+T} e^{-\left(r + \frac{(p^{ag})^2}{2}\right)(z-t) - p^{ag}(Z_z - Z_t)} e^{\left(\mu_A - \frac{\sigma_A^2}{2}\right)(z-t) + \sigma_A(Z_z - Z_t)} w_t \bar{l}_{i,z} dz \right] \\
&= \mathbb{E}_t \left[ \int_t^{s_i+T} e^{-\left(r + \frac{(p^{ag})^2 + \sigma_A^2}{2}\right)(z-t) + (\sigma_A - p^{ag})(Z_z - Z_t)} w_t e^{\mu_A(z-t)} \bar{l}_{i,z} dz \right] \\
&= \int_t^{s_i+T} e^{-(r+p^{ag}\sigma_A)(z-t)} w_t e^{\mu_A(z-t)} \bar{l}_{i,z} dz
\end{aligned}$$

which give us (16) by noticing that  $\mathbb{E}[w_z] = w_t e^{\mu_A(z-t)}$ .

Using the expression for the labor supply, we obtain

$$\begin{aligned}
h_{i,t} &= \sum_{i=1}^2 \beta_i e^{\gamma_i(t-s)} \int_t^{s_i+T} e^{-(r-\mu_A-\gamma_i+\eta^*\sigma_A)(z-t)} dz w_t \\
&= \sum_{i=1}^2 \beta_i e^{\gamma_i(t-s_i)} \frac{1 - e^{-(r-\mu_A-\gamma_i+p^{ag}\sigma_A)(T-(t-s_i))}}{r - \mu_A - \gamma_i + p^{ag}\sigma_A} w_t
\end{aligned}$$

Taking logs and differentiating, we obtain  $\sigma_{i,t}^h = \sigma_A$ .

**Solving the HJB equation:** The HJB equation for problem (9) is given by

$$\rho V_{i,t}(n_{i,t}; x_{i,t}) = \max_{c_{i,t}, k_{i,t}, l_{i,t}, \theta_{i,t}^{ag}, \theta_{i,t}^{id}} \left\{ \frac{c_{i,t}^{1-\gamma}}{1-\gamma} + \frac{\mathbb{E}_t [dV_{i,t}]}{dt} \right\}$$

subject to (7) and the boundary condition  $V_{i,s_i+T} = (1-\psi)\gamma V^* \frac{n_{i,s_i+T}^{1-\gamma}}{1-\gamma}$ .

The variable  $x_{i,t}$  is a state variable summarizing the evolution of the stochastic labor income  $w_t \bar{l}_{i,t}$ . A possibility would be to simply set  $x_t = w_{i,t}$ , but we will see below there is a more convenient alternative.

Using Ito's lemma, the HJB reduces to a partial differential equation for  $V_{i,t}(n_{i,t}; x_{i,t})$ :

$$\rho V_i = \max_{c_{i,t}, k_{i,t}, l_{i,t}, \theta_{i,t}^{ag}, \theta_{i,t}^{id}} \left\{ \frac{c_i^{1-\gamma}}{1-\gamma} + \frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial n_i} \mu_{n,i} + \frac{\partial V_i}{\partial x_i} \mu_{x,i} + \frac{1}{2} \frac{\partial^2 V_i}{\partial n_i^2} \sigma_{n,i}^2 + \frac{\partial^2 V_i}{\partial n_i \partial x_i} \sigma_{n,i} \sigma_{x,i} + \frac{1}{2} \frac{\partial V_i}{\partial x_i^2} \sigma_{x,i}^2 \right\}$$

subject to the same constraint and boundary condition, where the dependence on  $(t, n_i, x_i)$  is left implicit,  $(\mu_{n,i}, \sigma_{n,i})$  denote the drift and diffusion for  $n_{i,t}$ , and  $(\mu_{x,i}, \sigma_{x,i})$  denote the drift and diffusion for  $x_{i,t}$ .

First, we will set  $x_{i,t} = h_{i,t}$ . Since  $h_{i,t}$  is proportional to  $w_t$ , there is no loss in using  $h_{i,t}$  instead of  $w_t$ . We will guess-and-verify that the function  $V_{i,t}(n_{i,t}; h_{i,t}) = V_{s_i,t}^* \frac{(n_{i,t} + h_{i,t})^{1-\gamma}}{1-\gamma}$ , where  $V_{s_i,t}^*$  is a deterministic function of  $t$ , solves the PDE above.

Using the fact that the derivatives with respect to  $n_i$  and  $h_i$  are the same, we can write

$$\rho V_i = \max_{c_i, k_{i,t}, l_{i,t}, \theta_{i,t}^{ag}, \theta_{i,t}^{id}} \left\{ \frac{c_i^{1-\gamma}}{1-\gamma} + \frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial n_i} \mu_{\omega,i} + \frac{1}{2} \frac{\partial^2 V_i}{\partial n_i^2} \sigma_{\omega,i}^2 \right\}$$

where  $\omega_i = n_i + h_i$ ,  $\mu_{\omega,i} = \mu_{n,i} + \mu_{h,i}$ , and  $\sigma_{\omega,i}^2 = \sigma_{n,i}^2 + 2\sigma_{n,i}\sigma_{h,i} + \sigma_{h,i}^2$ .

Plugging into the HJB equation, we obtain

$$\begin{aligned} \frac{\rho}{1-\gamma} = & \max_{c_i, k_{i,t}, l_{i,t}, \theta_{i,t}^{ag}, \theta_{i,t}^{id}} \left\{ \frac{1}{1-\gamma} \frac{1}{V_{s,t}^*} \left( \frac{c_{i,t}}{\omega_{i,t}} \right)^{1-\gamma} + \frac{1}{1-\gamma} \frac{\dot{V}_{s,t}^*}{V_{s,t}^*} + r + \frac{q_t k_{i,t}}{\omega_{i,t}} (\mu_{i,t}^R - r - p^{ag} \theta_{i,t}^{ag}) + \frac{h_{i,t}}{\omega_{i,t}} \sigma_A p^{ag} \right. \\ & \left. - \frac{c_{i,t}}{\omega_{i,t}} - \frac{\gamma}{2} \left[ \left( \frac{q_t k_{i,t}}{\omega_{i,t}} (\sigma_A - \theta_{i,t}^{ag}) + \frac{h_{i,t}}{\omega_{i,t}} \sigma_A \right)^2 + \left( \frac{q_t k_{i,t}}{\omega_{i,t}} (\sigma_{id} - \theta_{i,t}^{id}) \right)^2 \right] \right\} \end{aligned}$$

From the expression above, it is immediate that the optimal value of  $(l_{i,t}, \theta_{i,t}^{id})$  maximizes the expected return on the business. The first-order conditions for  $(l_{i,t}, \theta_{i,t}^{id})$  are given in (10) and (11), respectively. The expected return on the business will be equalized, allowing us to write  $\mu_{i,t}^R = \mu^R$ .

Optimal quantity of capital, aggregate insurance, and idiosyncratic insurance solve the problem

$$\max_{k_{i,t}, \theta_{i,t}^{ag}, \theta_{i,t}^{id}} \left\{ \frac{q_t k_{i,t}}{\omega_{i,t}} (\mu^R - r - p^{ag} \theta_{i,t}^{ag}) + \frac{h_{i,t}}{\omega_{i,t}} \sigma_A p^{ag} - \frac{\gamma}{2} \left[ \left( \frac{q_t k_{i,t}}{\omega_{i,t}} (\sigma_A - \theta_{i,t}^{ag}) + \frac{h_{i,t}}{\omega_{i,t}} \sigma_A \right)^2 + \left( \frac{q_t k_{i,t}}{\omega_{i,t}} (\sigma_{id} - \theta_{i,t}^{id}) \right)^2 \right] \right\}$$

Multiplying the expression above by  $\omega_{i,t}/a_{i,t}$  gives (17). The first order condition for  $\theta_{i,t}^{id}$

$$\gamma \frac{q_t k_{i,t}}{\omega_{i,t}} (\sigma_{id} - \theta_{i,t}^{id}) = p_{i,t}^{id} \quad (47)$$

where  $\frac{q_t k_{i,t}}{\omega_{i,t}} p_{i,t}^{id}$  denotes the Lagrange multiplier on the skin-in-the-game constraint.

The first-order conditions for the amount of capital and aggregate insurance can be written as

$$\begin{aligned} \mu^R - r - p^{ag} \theta_{i,t} &= \gamma \left[ \left( \frac{q_t k_{i,t}}{\omega_{i,t}} (\sigma_A - \theta_{i,t}^{ag}) + \frac{h_{i,t}}{\omega_{i,t}} \sigma_A \right) (\sigma_A - \theta_{i,t}^{ag}) + \frac{q_t k_{i,t}}{\omega_{i,t}} (\sigma_{id} - \theta_{i,t}^{id})^2 \right] \\ p^{ag} &= \gamma \left( \frac{q_t k_{i,t}}{\omega_{i,t}} (\sigma_A - \theta_{i,t}^{ag}) + \frac{h_{i,t}}{\omega_{i,t}} \sigma_A \right) \end{aligned}$$

Combining the expressions above, we obtain

$$\frac{q_t k_{i,t}}{\omega_{i,t}} = \frac{\mu^R - r - p^{ag} \sigma_A}{\gamma (\sigma_{id} - \theta_{i,t}^{id})^2} \quad (48)$$

If  $\mu^R - r - p^{ag} \sigma_A > 0$ , then  $k_{i,t} > 0$  and  $p_{i,t}^{id} > 0$ , so  $\theta_{i,t}^{id} = (1 - \phi) \sigma_{id}$ . The Lagrange multiplier  $p_{i,t}^{id}$  is then given by

$$p_{i,t}^{id} = \frac{\mu^R - r - p^{ag} \sigma_A}{\phi \sigma_{id}} \quad (49)$$

which coincides with expression (18) after we write  $p_{i,t} = p^{id}$ .

The demand for capital can then be written as

$$\frac{q_t k_{i,t}}{\omega_{i,t}} = \frac{p^{id}}{\gamma \phi \sigma_{id}} \quad (50)$$

Multiplying by  $\omega_{i,t}/a_{i,t}$ , we obtain expression (19). Solving for  $\theta_{i,t}^{ag}$  in the optimality condition for aggregate insurance we obtain (20).

The first-order condition for consumption gives the consumption-total-wealth ratio will satisfy

$$\frac{c_{i,t}}{\omega_{i,t}} = (V_{s_i,t}^*)^{-\frac{1}{\gamma}} \quad (51)$$

Plugging the expressions above back into the HJB, we obtain a differential equation for  $\bar{V}_{s,t}$

$$\frac{\dot{V}_{s_i,t}^*}{V_{s_i,t}^*} + \gamma (V_{s_i,t}^*)^{-\frac{1}{\gamma}} = \rho + (\gamma - 1) \left[ r + \frac{(p^{ag})^2 + (p^{ag})^2}{2\gamma} \right]$$

with boundary condition  $V_{s_i,s_i+T}^* = (1 - \psi)^\gamma V^*$ .

Define  $z_{s_i,t} \equiv (V_{s_i,t}^*)^{\frac{1}{\gamma}}$  and  $\bar{r} = \frac{1}{\gamma} \rho + \left(1 - \frac{1}{\gamma}\right) \left[ r + \frac{(p^{id})^2 + (p^{ag})^2}{2\gamma} \right]$  to obtain

$$\dot{z}_{s_i,t} - \bar{r} z_{s_i,t} = -1 \quad (52)$$

Solving the differential equation, we obtain

$$(V_{s_i,t}^*)^{\frac{1}{\gamma}} = \frac{1 - e^{-\bar{r}(T-(t-s_i))}}{\bar{r}} + e^{-\bar{r}(T-(t-s_i))} (1 - \psi) (V^*)^{\frac{1}{\gamma}} \quad (53)$$

We will assume that  $(V^*)^{\frac{1}{\gamma}} = \frac{1}{\bar{r}}$ , so the value function coefficient for  $\psi = 0$  will coincide with the one for an infinite horizon agent. The consumption-wealth ratio is then given by

$$(V_{s_i,t}^*)^{-\frac{1}{\gamma}} = \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-(t-s_i))}} \quad (54)$$

The consumption-wealth ratio can then be written as

$$\frac{c_{i,t}}{n_{i,t}} = \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-(t-s_i))}} \left( 1 + \frac{h_{i,t}}{n_{i,t}} \right) \quad (55)$$

which coincides with (21). □

## B.2 Proof of proposition 2

*Proof. Derivation of (118):* The share of total wealth held by an entrepreneur with age  $a$  at date  $t$  is denoted by  $\tilde{\omega}_{t,a} = \omega_{t-a,t} / \omega_{e,t}$ . In a stationary equilibrium,  $\tilde{\omega}_{t,a} = \tilde{\omega}_a$ , i.e., it is independent of calendar

time  $t$ .

$$\tilde{\omega}_a = \frac{\omega_{t-a,t}}{\omega_{e,t}} = \frac{\omega_{s,s+a}}{\omega_{e,s+a}} \quad (56)$$

by setting  $t = s + a$ .

The law of motion of  $\omega_{s,t}$  is given by

$$\frac{d\omega_{s,t}}{\omega_{s,t}} = \left[ r + \gamma\sigma_A^2 + \frac{(p^{id})^2}{\gamma} - \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-(t-s))}} \right] dt + \sigma_A dZ_t \quad (57)$$

using the fact that the idiosyncratic shocks averaged out by an exact law of large numbers.

From the demand for capital, we have

$$q_t k_{i,t} = \frac{p^{id}}{\gamma\phi\sigma_{id}} \omega_{i,t} \Rightarrow q_t k_t = \frac{p^{id}}{\gamma\phi\sigma_{id}} \omega_{e,t} \quad (58)$$

The value of  $\omega_{e,t}$  will then grow at the same rate as the aggregate value of capital

$$\frac{d\omega_{e,t}}{\omega_{e,t}} = (g + \mu_A) dt + \sigma_{A,t} dZ_t \quad (59)$$

Applying Ito's lemma to  $\tilde{\omega}_a$ :

$$\begin{aligned} \frac{d\tilde{\omega}_a}{\tilde{\omega}_a} &= \frac{d\omega_{s,s+a}}{\omega_{s,s+a}} - \frac{d\omega_{e,s+a}}{\omega_{e,s+a}} + \left( \frac{d\omega_{e,s+a}}{\omega_{e,s+a}} \right)^2 - \frac{d\omega_{s,s+a}}{\omega_{s,s+a}} \frac{d\omega_{e,s+a}}{\omega_{e,s+a}} \\ &= \left[ r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}} \right] da \end{aligned}$$

Integrating the expression above, we obtain

$$\log \tilde{\omega}_a = \log \tilde{\omega}_0 + \left[ r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - \bar{r} \right] a + \log \left( \frac{1 - \psi e^{-\bar{r}(T-a)}}{1 - \psi e^{-\bar{r}T}} \right) \quad (60)$$

Exponentiating the previous expression gives us

$$\tilde{\omega}_a = \tilde{\omega}_0 e^{\left( r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - \bar{r} \right) a} \frac{1 - \psi e^{-\bar{r}(T-a)}}{1 - \psi e^{-\bar{r}T}} \quad (61)$$

The share of wealth of the oldest households is given by

$$\tilde{\omega}_T = \tilde{\omega}_0 e^{\left( r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - \bar{r} \right) T} \frac{1 - \psi}{1 - \psi e^{-\bar{r}T}} \quad (62)$$

Since  $\tilde{\omega}_T = \tilde{n}_0$  and  $\tilde{\omega}_0 = \tilde{n}_0 + \tilde{h}_0$ , then

$$\tilde{n}_0 = \frac{e^{\left( r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - mpc_e \right) T}}{1 - e^{\left( r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - mpc_e \right) T}} \tilde{h}_0 \quad (63)$$

where  $mpc_e = \frac{1}{T} \int_0^T \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}} da$  is the average MPC.

Finally, since  $\tilde{\omega}_a = \tilde{n}_a + \tilde{h}_a = \tilde{n}_a \left(1 + \frac{h_a}{n_a}\right)$ , then

$$\log \tilde{n}_a = \log \tilde{n}_0 + \log \left( \frac{1 + \frac{h_0}{n_0}}{1 + \frac{h_a}{n_a}} \right) + \left[ r + \frac{(p^{ng})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - \bar{r} \right] a + \log \left( \frac{1 - \psi e^{-\bar{r}(T-a)}}{1 - \psi e^{-\bar{r}T}} \right) \quad (64)$$

Similarly, the law of motion of  $\omega_{e,t} = \frac{g e^{-gt}}{1 - e^{-gT}} \int_{t-T}^t e^{gs} \omega_{s,t} ds$  is given by

$$\begin{aligned} \frac{d\omega_{e,t}}{\omega_{e,t}} &= \frac{g e^{-gt}}{1 - e^{-gT}} \int_{t-T}^t e^{gs} \frac{\omega_{s,t}}{\omega_{e,t}} \frac{d\omega_{s,t}}{\omega_{s,t}} ds + \left[ \frac{g e^{-gt}}{1 - e^{-gT}} \left( \frac{e^{gt} \omega_{t,t} - e^{g(t-T)} \omega_{t-T,t}}{\omega_{e,t}} \right) - g \right] dt \\ &= \left[ r + \gamma \sigma_A^2 + \frac{(p^{id})^2}{\gamma} - \int_{t-T}^t \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-(t-s))}} \frac{g e^{-g(t-s)}}{1 - e^{-gT}} \frac{\omega_{s,t}}{\omega_{e,t}} ds + \frac{g}{1 - e^{-gT}} \frac{h_{t,t}}{\omega_{e,t}} - g \right] dt + \sigma_A dZ_t \end{aligned}$$

using the fact that  $e^{gt} \omega_{t,t} - e^{g(t-T)} \omega_{t-T,t} = e^{gt} h_{t,t}$ .

The law of motion of  $\tilde{\omega}_{s,t} = \omega_{s,t} / \omega_{e,t}$  is given by

$$\begin{aligned} \frac{d\tilde{\omega}_{s,t}}{\tilde{\omega}_{s,t}} &= \frac{\omega_{s,t}}{\omega_{s,t}} - \frac{d\omega_{e,t}}{\omega_{e,t}} + \left( \frac{d\omega_{e,t}}{\omega_{e,t}} \right)^2 - \frac{d\omega_{s,t}}{\omega_{s,t}} \frac{\omega_{e,t}}{\omega_{e,t}} \\ &= \left[ \int_0^T \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}} \frac{g e^{-ga}}{1 - e^{-gT}} \tilde{\omega}_{t-a,t} da - \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-(t-s))}} - \frac{g}{1 - e^{-gT}} \frac{h_{t,t}}{\omega_{e,t}} + g \right] dt \end{aligned}$$

Hence,  $\tilde{\omega}_{s,t}$  is non-stochastic. Let  $a = t - s$ , so we can write  $\tilde{\omega}_{s,t} = \tilde{\omega}_{s,s+a}$ . We will focus on a stationary equilibrium where  $\tilde{\omega}_{s,s+a}$  does not depend on  $s$ . We will abuse notation and write  $\tilde{\omega}_{s,s+a} = \tilde{\omega}_a$ . Integrating the equation above, we obtain

$$\log \tilde{\omega}_a = \log \tilde{\omega}_0 - \int_0^a (mpc_{\bar{a}} - mpc_e) d\tilde{a} - [f(0)(\tilde{\omega}_0 - 1) - f(T)(\tilde{\omega}_T - 1)] a \quad (65)$$

Notice that  $\omega_{s,t} = n_{s,t} \left(1 + \frac{h_{s,t}}{n_{s,t}}\right)$  and  $\omega_{e,t} = n_{e,t} \left(1 + \frac{h_{e,t}}{n_{e,t}}\right)$ , so  $\tilde{\omega}_a = \tilde{n}_a \frac{1 + \frac{h_a}{n_a}}{1 + \frac{h_e}{n_e}}$ . This allow us to write the expression above as

$$\log \tilde{n}_a = \log \tilde{n}_0 + \log \left( \frac{1 + \frac{h_0}{n_0}}{1 + \frac{h_a}{n_a}} \right) - \int_0^a (mpc_{\bar{a}} - mpc_e) d\tilde{a} - [f(0)(\tilde{\omega}_0 - 1) - f(T)(\tilde{\omega}_T - 1)] a \quad (66)$$

**Computation:**

$$\frac{h_a}{n_a} = \frac{h_a}{\omega_a - h_a} = \frac{h_a/h_0}{\omega_a/h_0 - h_a/h_0} \quad (67)$$

$$1 + \frac{h_0}{n_0} = \frac{\omega_0/h_0}{\omega_0/h_0 - 1} = \frac{1}{1 - h_0/\omega_0} \quad (68)$$

where

$$\frac{\omega_0}{h_0} = \frac{1}{1 - e^{\left(r + \frac{(p^a g)^2}{\gamma} + \frac{(p^i d)^2}{\gamma} - (g + \mu_A) - \bar{r}\right)T} \frac{1 - \psi}{1 - \psi e^{-\bar{r}T}}} \quad (69)$$

Then,

$$1 + \frac{h_0}{n_0} = e^{-\left(r + \frac{(p^a g)^2}{\gamma} + \frac{(p^i d)^2}{\gamma} - (g + \mu_A) - \bar{r}\right)T} \frac{1 - \psi e^{-\bar{r}T}}{1 - \psi} \quad (70)$$

$$\omega_a = \frac{e^{\left[r + \frac{(p^a g)^2}{\gamma} + \frac{(p^i d)^2}{\gamma} - \mu_A - \bar{r}\right]a} \frac{1 - \psi e^{-\bar{r}(T-a)}}{1 - \psi e^{-\bar{r}T}} \frac{h_0}{1 - e^{\left(r + \frac{(p^a g)^2}{\gamma} + \frac{(p^i d)^2}{\gamma} - (g + \mu_A) - \bar{r}\right)T} \frac{1 - \psi}{1 - \psi e^{-\bar{r}T}}} \omega_e \quad (71)$$

Solving explicitly for  $\tilde{\omega}_a$ , we obtain

$$\tilde{\omega}_a = \tilde{\omega}_0 \frac{e^{(\zeta - \bar{r})a} - \psi e^{-\bar{r}T} e^{\zeta a}}{1 - \psi e^{-\bar{r}T}} \quad (72)$$

where

$$\zeta = \int_0^T \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}} \frac{g e^{-g a}}{1 - e^{-g T}} \tilde{\omega}_a da - \frac{g}{1 - e^{-g T}} \frac{h_{t,t}}{\omega_{e,t}} + g \quad (73)$$

Combining the previous two equations, we obtain an expression for  $\zeta$ :

$$\zeta - g = \frac{g}{1 - e^{-g T}} \left[ \tilde{\omega}_0 \frac{\bar{r}}{1 - \psi e^{-\bar{r}T}} \frac{e^{(\zeta - g - \bar{r})T} - 1}{\zeta - g - \bar{r}} - \tilde{h}_0 \right] \quad (74)$$

The value of  $\tilde{\omega}_0$  can be obtained by using the fact that  $\tilde{\omega}_a f(a)$  integrates to one:

$$1 = \frac{\tilde{\omega}_0}{1 - \psi e^{-\bar{r}T}} \frac{g}{1 - e^{-g T}} \left[ \frac{e^{(\zeta - g - \bar{r})T} - 1}{\zeta - g - \bar{r}} - \psi e^{-\bar{r}T} \frac{e^{(\zeta - g)T} - 1}{\zeta - g} \right] \quad (75)$$

Hence, the expression for  $\zeta$  can be written as

$$\zeta - g = \frac{\bar{r}}{1 - \psi e^{-\bar{r}T} \frac{e^{(\zeta - g)T} - 1}{\frac{e^{(\zeta - g - \bar{r})T} - 1}{\zeta - g - \bar{r}}}} - \frac{g}{1 - e^{-g T}} \tilde{h}_0 \quad (76)$$

□

### B.3 Proof of lemma 2

We will derive the Kolmogorov Forward Equation as the limit of a discrete time approximation. The law of motion of wealth is given by

$$d\tilde{n}_{i,t} = \mu_{\tilde{n}}(\tilde{n}_{i,t}, a_i) dt + \sigma_{\tilde{n}}(\tilde{n}_{i,t}, a_i) dZ_{i,t} \quad (77)$$

The discrete time approximation goes as follows. Time takes values on the discrete set  $\{t_1, \dots, t_L\}$ ,



where  $\Delta t = t_{l+1} - t_l$  is the constant time step. Normalized wealth  $\tilde{n}_{i,t}$  will take values in a discrete grid,  $\tilde{n}_{i,t} \in \{\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_J\}$  with a constant step size  $\Delta\tilde{n} = \tilde{n}_{j+1} - \tilde{n}_j$ . Age is also assumed to take values in a discrete grid  $\{a_1, \dots, a_K\}$ , where  $\Delta a = a_{k+1} - a_k$ . The probability of moving up, down, or staying at the same point of the grid are chosen to approximate (77) and are given, respectively, by

$$\begin{aligned} p_u(\tilde{n}_j, a_k) &= \frac{1}{2} \left[ \frac{\sigma_{\tilde{n}}(\tilde{n}_j, a_k)^2}{\bar{\sigma}^2} + \frac{\mu_{\tilde{n}}(\tilde{n}_j, a_k)}{\bar{\sigma}^2} \Delta\tilde{n} \right] \\ p_d(\tilde{n}_j, a_k) &= \frac{1}{2} \left[ \frac{\sigma_{\tilde{n}}(\tilde{n}_j, a_k)^2}{\bar{\sigma}^2} - \frac{\mu_{\tilde{n}}(\tilde{n}_j, a_k)}{\bar{\sigma}^2} \Delta\tilde{n} \right] \\ p_s(\tilde{n}_j, a_k) &= 1 - \frac{1}{2} \frac{\sigma_{\tilde{n}}(\tilde{n}_j, a_k)^2}{\bar{\sigma}^2} \end{aligned}$$

where  $\bar{\sigma} = \max_{1 \leq j \leq J, 1 \leq k \leq K} \sigma_{\tilde{n}}(\tilde{n}_j, a_k)$ ,  $\Delta\tilde{n} = \bar{\sigma} \sqrt{\Delta t}$ , and  $\Delta a = \Delta t$ .

Notice that the expected change in  $\tilde{n}_{i,t}$ , where  $\tilde{n}_{i,t} = \tilde{n}_j$  and  $a_i = a_k$ , is given by

$$\mathbb{E} [\tilde{n}_{i,t+1} - \tilde{n}_{i,t}] = p_u(\tilde{n}_j, a_k) \Delta\tilde{n} + p_d(\tilde{n}_j, a_k) (-\Delta\tilde{n}) = \mu_{\tilde{n}}(\tilde{n}_j, a_k) \Delta t \quad (78)$$

and

$$\mathbb{E} [(\tilde{n}_{i,t+1} - \tilde{n}_{i,t})^2] = p_u(\tilde{n}_j, a_k) \Delta\tilde{n}^2 + p_d(\tilde{n}_j, a_k) (-\Delta\tilde{n})^2 = \sigma_{\tilde{n}}(\tilde{n}_j, a_k)^2 \Delta t \quad (79)$$

Let  $f(\tilde{n}_j, a_k, t_l)$  denote the mass of agents with normalized wealth  $\tilde{n}_j$ , age  $a_k$ , at period  $t_l$ . The law of motion of  $f$  is given by

$$\begin{aligned} f(\tilde{n}_j, a_k, t_l + \Delta t) &= p_u(\tilde{n}_j - \Delta\tilde{n}, a_k - \Delta a) f(\tilde{n}_j - \Delta\tilde{n}, a_k - \Delta a, t_l) + p_s(\tilde{n}_j, a_k) f(\tilde{n}_j, a_k - \Delta a, t_l) \\ &\quad + p_d(\tilde{n}_j + \Delta\tilde{n}, a_k - \Delta a) f(\tilde{n}_j + \Delta\tilde{n}, a_k - \Delta a, t_l) \end{aligned}$$

Taking a Taylor expansion of both sides, we obtain

$$\begin{aligned} f + f_t \Delta t &= \frac{1}{2} \left( \frac{\sigma_{\tilde{n}}^2 - (\sigma_{\tilde{n}}^2)_{\tilde{n}} \Delta\tilde{n} + 0.5 (\sigma_{\tilde{n}}^2)_{\tilde{n}\tilde{n}} \Delta\tilde{n}^2}{\bar{\sigma}^2} + \frac{\mu_{\tilde{n}} - (\mu_{\tilde{n}})_{\tilde{n}} \Delta\tilde{n}}{\bar{\sigma}^2} \Delta\tilde{n} \right) (f - f_a \Delta t - f_{\tilde{n}} \Delta\tilde{n} + 0.5 f_{\tilde{n}\tilde{n}} \Delta\tilde{n}^2) \\ &\quad + \frac{1}{2} \left( \frac{\sigma_{\tilde{n}}^2 + (\sigma_{\tilde{n}}^2)_{\tilde{n}} \Delta\tilde{n} + 0.5 (\sigma_{\tilde{n}}^2)_{\tilde{n}\tilde{n}} \Delta\tilde{n}^2}{\bar{\sigma}^2} - \frac{\mu_{\tilde{n}} + (\mu_{\tilde{n}})_{\tilde{n}} \Delta\tilde{n}}{\bar{\sigma}^2} \Delta\tilde{n} \right) (f - f_a \Delta t + f_{\tilde{n}} \Delta\tilde{n} + 0.5 f_{\tilde{n}\tilde{n}} \Delta\tilde{n}^2) \\ &\quad + \left( 1 - \frac{\sigma_{\tilde{n}}^2}{\bar{\sigma}^2} \right) (f - f_a \Delta t) + o(\Delta t) \end{aligned}$$

Simplifying the expression above and taking the limit  $\Delta t \rightarrow 0$ , we obtain

$$f_t + f_a = \frac{1}{2} (\sigma_{\tilde{n}}^2)_{\tilde{n}\tilde{n}} f - (\mu_{\tilde{n}})_{\tilde{n}} f + (\sigma_{\tilde{n}}^2)_{\tilde{n}} f_{\tilde{n}} - \mu_{\tilde{n}} f_{\tilde{n}} + (\sigma_{\tilde{n}^2})_{\tilde{n}} f_{\tilde{n}} + \frac{1}{2} \sigma_{\tilde{n}}^2 f_{\tilde{n}\tilde{n}} \quad (80)$$

or more explicitly

$$\frac{\partial f(\tilde{n}, a, t)}{\partial t} + \frac{\partial f(\tilde{n}, a, t)}{\partial a} = - \frac{\partial [f(\tilde{n}, a, t) \mu_{\tilde{n}}(\tilde{n}, a)]}{\partial \tilde{n}} + \frac{1}{2} \frac{\partial [f(\tilde{n}, a, t) \sigma_{\tilde{n}}^2(\tilde{n}, a)]}{\partial \tilde{n}^2} \quad (81)$$

In a stationary equilibrium, we can ignore the dependence on calendar time to obtain

$$\frac{\partial f(\tilde{n}, a)}{\partial a} = -\frac{\partial [f(\tilde{n}, a)\mu_{\tilde{n}}(\tilde{n}, a)]}{\partial \tilde{n}} + \frac{1}{2} \frac{\partial [f(\tilde{n}, a)\sigma_{\tilde{n}}^2(\tilde{n}, a)]}{\partial \tilde{n}^2} \quad (82)$$

Let  $f(\tilde{n}|a)$  denote the conditional density, so  $f(\tilde{n}, a) = f(\tilde{n}|a)f(a)$ . We can write the Kolmogorov Forward Equation in terms of the conditional density:

$$f(a) \frac{\partial f(\tilde{n}|a)}{\partial a} + f(n|a)f'(a) = -f(a) \frac{\partial [f(\tilde{n}|a)\mu_{\tilde{n}}(\tilde{n}, a)]}{\partial \tilde{n}} + f(a) \frac{1}{2} \frac{\partial [f(\tilde{n}|a)\sigma_{\tilde{n}}^2(\tilde{n}, a)]}{\partial \tilde{n}^2} \quad (83)$$

Dividing by  $f(a)$  and using the fact that  $f'(a) = -gf(a)$ , we obtain

$$\frac{\partial f(\tilde{n}|a)}{\partial a} = -\frac{\partial [f(\tilde{n}|a)\mu_{\tilde{n}}(\tilde{n}, a)]}{\partial \tilde{n}} + \frac{1}{2} \frac{\partial [f(\tilde{n}|a)\sigma_{\tilde{n}}^2(\tilde{n}, a)]}{\partial \tilde{n}^2} + gf(n|a) \quad (84)$$

In the case entrepreneurs leave some bequests,  $0 \leq \psi < 1$ , the PDE is subject to the boundary condition  $f(\tilde{n}|0) = f(\tilde{n}|T)$ . In the absence of bequests,  $\psi = 1$ , the PDE is subject to the boundary condition  $f(\tilde{n}|0) = \delta(\tilde{n})$ , where  $\delta(\cdot)$  is the Dirac delta function.

Let  $F(\tilde{n}, t; s)$  denote the CDF of normalized wealth at date  $t$  for entrepreneurs born at date  $s$ . We can write

$$F(\tilde{n}, t, s) = \mathbb{E}[g(\tilde{n}_{i,t})|s_i = s] \quad (85)$$

where  $g(n) = \mathbf{1}_{n \leq \tilde{n}}$ .

$$\frac{\partial f(\tilde{\omega}|a)}{\partial a} = -\frac{\partial [f(\tilde{\omega}|a)\mu_{\tilde{\omega}}(\tilde{\omega}, a)]}{\partial \tilde{\omega}} + \frac{1}{2} \frac{\partial^2 [f(\tilde{\omega}|a)\sigma_{\tilde{\omega}}^2(\tilde{\omega}, a)]}{\partial \tilde{\omega}^2} \quad (86)$$

$$f(n|a) = \frac{1}{\tilde{n} + \tilde{h}_a} \frac{1}{\sigma_a \sqrt{2\pi}} \exp\left(-\frac{(\log(\tilde{n} + \tilde{h}_a) - \mu_a)^2}{2\sigma_a^2}\right) \quad (87)$$

## B.4 Proof of proposition 3

The law of motion of total wealth

$$d \log \tilde{\omega}_{i,t} = \left[ r + \frac{(p^{as})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-(t-s))}} - \mu_A - \frac{1}{2} \left( \frac{p^{id}}{\gamma} \right)^2 \right] dt + \frac{p^{id}}{\gamma} dZ_{i,t}$$

where  $s_i = s$ .

Integrating the expression above, we obtain

$$\log \tilde{\omega}_{i,t} = \log \tilde{\omega}_{i,s} + \int_s^t \left[ r + \frac{(p^{as})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \frac{\bar{r}}{1 - e^{-\bar{r}(T-(\tilde{t}-s))}} - \mu_A - \frac{1}{2} \left( \frac{p^{id}}{\gamma} \right)^2 \right] d\tilde{t} + \frac{p^{id}}{\gamma} (Z_{i,t} - Z_{i,s})$$

where  $Z_{i,t} - Z_{i,s} \sim \mathcal{N}(0, a^2)$ ,  $a = t - s$ .

Hence,  $\log \tilde{\omega}_{i,t} \sim \mathcal{N}(m_a, v_a)$ , where

$$m_a = \log \tilde{h}_0 + \left[ r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A - \frac{1}{2} \left( \frac{p^{id}}{\gamma} \right)^2 - \bar{r} \right] a + \log \frac{1 - e^{-\bar{r}(T-a)}}{1 - e^{-\bar{r}T}}$$

$$v_a = \left( \frac{p^{id}}{\gamma} \right)^2 a$$

Normalized financial wealth  $\tilde{n}_{i,t} = \tilde{\omega}_{i,t} - \tilde{h}_{i,t}$  has a shifted log-normal distribution conditional on  $s_i = s$ , with support  $(-\tilde{h}_{s,t}, \infty)$ . The expected value and variance of  $\tilde{n}_{i,t}$  is given by

$$\mathbb{E}[\tilde{n}_{i,t} | a_i = a] = \tilde{h}_0 e^{\left( r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A - \bar{r} \right) a} \frac{1 - e^{-\bar{r}(T-a)}}{1 - e^{-\bar{r}T}} - \tilde{h}_{i,t}$$

$$\mathbb{V}[\tilde{n}_{i,t} | a_i = a] = \left[ e^{\left( \frac{p^{id}}{\gamma} \right)^2 a} - 1 \right] \left[ \tilde{h}_0 e^{\left( r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A \right) a} \frac{e^{-\bar{r}a} - e^{-\bar{r}T}}{1 - e^{-\bar{r}T}} \right]^2$$

Let's now show that  $\mathbb{V}[\tilde{n}_{i,t} | a_i = a]$  has an inverted U shape. Define the following functions:

$$f(a) = \left[ e^{\left( \frac{p^{id}}{\gamma} \right)^2 a} - 1 \right]^{\frac{1}{2}} e^{\left( r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A \right) a}; \quad g(a) = \frac{e^{-\bar{r}a} - e^{-\bar{r}T}}{1 - e^{-\bar{r}T}} \quad (88)$$

The derivative of the product of  $f(a)$  and  $g(a)$  will be positive if

$$f'(a)g(a) + f(a)g'(a) > 0 \iff \frac{f'(a)}{f(a)} + \frac{g'(a)}{g(a)} > 0 \quad (89)$$

for  $a \neq 0$  and  $a \neq T$ .

Notice that  $g'(a)/g(a)$  is negative, monotonically decreasing, and approaches  $-\infty$  as  $a$  approaches  $T$ :

$$\frac{g'(a)}{g(a)} = -\bar{r} \frac{1}{1 - e^{-\bar{r}(T-a)}} \quad (90)$$

The term  $f'(a)/f(a)$  is positive, monotonically decreasing, and approaches  $+\infty$  as  $a$  approaches 0:

$$\frac{f'(a)}{f(a)} = \frac{1}{2} \left( \frac{p^{id}}{\gamma} \right)^2 \frac{e^{\left( \frac{p^{id}}{\gamma} \right)^2 a}}{e^{\left( \frac{p^{id}}{\gamma} \right)^2 a} - 1} + r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A \quad (91)$$

Hence, there exists a unique  $0 < \hat{a} < T$  such that  $f'(a)g(a) + f(a)g'(a) > 0$  for all  $a < \hat{a}$  and  $f'(a)g(a) + f(a)g'(a) < 0$  for all  $a > \hat{a}$ . Hence,  $\mathbb{V}[\tilde{n}_{i,t} | a_i = a]$  is equal to zero at  $a = 0$ , it increases monotonically for  $a < \hat{a}$ , where it achieves the maximum, and it decreases towards zero for  $\hat{a} < a \leq T$ .

Let  $f(\tilde{n}, t; 0, s)$  denote the density at date  $t$  of normalized wealth  $\tilde{n}$  of entrepreneurs who were born at date  $s$  with zero financial wealth.

## B.5 Derivation of the value of human wealth

First, consider the value of the financier's human wealth

$$h_{f,t} = \int_t^\infty e^{-(r+p^a g \sigma_A - g - \mu_A)(z-t)} w_t e^{g t} \frac{1 - e^{-g T}}{g} \bar{l}_f dz = \frac{1 - e^{-g T}}{g} \frac{w_t e^{g t} \bar{l}_f}{r + p^a g \sigma_A - (g + \mu_A)}$$

using the fact that  $\bar{l}_{f,t} = e^{g t} \frac{1 - e^{-g T}}{g} \bar{l}_f$ .

Hence, financier's human wealth can be written as  $h_{f,t} = A_t e^{g t} \frac{1 - e^{-g T}}{g} h_f$ , where

$$h_f = \frac{\bar{l}_f}{r + p^a g \sigma_A - (g + \mu_A)} (1 - \alpha) k^\alpha \quad (92)$$

The value of human wealth at date  $t$  for an entrepreneur born at date  $s$  is given by

$$\begin{aligned} h_{s,t} &= \int_t^{s+T} e^{-(r+p^a g \sigma_A - \mu_A)(z-t)} \sum_{k=1}^K \Gamma_k e^{\varphi_k(z-s)} w_t \bar{l}_e dz \\ &= \sum_{k=1}^K \Gamma_k e^{\varphi_k(t-s)} \int_t^{s+T} e^{-(r+p^a g \sigma_A - \mu_A - \varphi_k)(z-t)} dz w_t \bar{l}_e \\ &= \sum_{k=1}^K \Gamma_k e^{\varphi_k(t-s)} \frac{1 - e^{-(r+p^a g \sigma_A - \mu_A - \varphi_k)(T-(t-s))}}{r + p^a g \sigma_A - \mu_A - \varphi_k} w_t \bar{l}_e \end{aligned}$$

Hence, the expression above can be written as  $h_{s,t} = A_t h_{t-s}$ , where  $h_a$  is given by

$$h_a = \sum_{k=1}^K \Gamma_k e^{\varphi_k a} \frac{1 - e^{-(r+p^a g \sigma_A - \mu_A - \varphi_k)(T-a)}}{r + p^a g \sigma_A - \mu_A - \varphi_k} \bar{l}_e (1 - \alpha) k^\alpha \quad (93)$$

Total wealth in the economy can be written  $h_t$  is given by

$$\begin{aligned} h_t &= \int_{t-T}^t e^{g s} h_{s,t} ds + h_{f,t} \\ &= \left[ \int_0^T \frac{g e^{-g a}}{1 - e^{-g T}} h_a da + h_{f,t} \right] A_t e^{g t} \frac{1 - e^{-g T}}{g} \end{aligned}$$

Hence, human wealth can be written as  $h_t = A_t e^{g t} \frac{1 - e^{-g T}}{g} h$ , where

$$h = \left[ \int_0^T \sum_{k=1}^K \Gamma_k \frac{g e^{-g a}}{1 - e^{-g T}} e^{\varphi_k a} \frac{1 - e^{-(r+p^a g \sigma_A - \mu_A - \varphi_k)(T-a)}}{r + p^a g \sigma_A - \mu_A - \varphi_k} \bar{l}_e da + \frac{\bar{l}_f}{r + p^a g \sigma_A - (g + \mu_A)} \right] (1 - \alpha) k^\alpha \quad (94)$$

where  $\bar{l}_f = 1 - \bar{l}_e$ .

## B.6 Derivation of the expression for $x_e$

First, let's derive how total wealth is distributed among entrepreneurs. The share of financial wealth held by entrepreneurs of age  $a$  is given by  $n_a = f(a)n_{t-a,t}/n_{e,t}$ . Analogously, define the share of total wealth held by entrepreneurs as  $\omega_a = f(a)\omega_{t-a,t}/\omega_{e,t}$ . Integrating the expression for  $a$ , we obtain

$$n_a + h_a = (n_0 + h_0)e^{\left[r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A)\right]a} \frac{\psi e^{-\bar{r}a} - \psi e^{-\bar{r}T}}{1 - \psi e^{-\bar{r}T}} \quad (95)$$

where

$$n_0 + h_0 = \frac{1}{1 - e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A)\right)T} \frac{(1-\psi)e^{-\bar{r}T}}{1 - \psi e^{-\bar{r}T}}} h_0 \quad (96)$$

Combining the two equations, we obtain

$$\omega_a = \frac{e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - \bar{r}\right)a} \frac{1 - \psi e^{-\bar{r}(T-a)}}{1 - \psi e^{-\bar{r}T}} \hat{h}_0}{1 - e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - \bar{r}\right)T} \frac{1 - \psi}{1 - \psi e^{-\bar{r}T}}} x_e \quad (97)$$

where  $\hat{h}_0 \equiv f(0)h_a/(qk + h)$ .

Notice that  $\bar{r}_f = r + \frac{(p^{ag})^2}{\gamma} - (g + \mu_A)$ , so the expression above can be written as

$$\omega_a = \frac{e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)a} \frac{1 - \psi e^{-\bar{r}(T-a)}}{1 - \psi e^{-\bar{r}T}} \hat{h}_0}{1 - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T} \frac{1 - \psi}{1 - \psi e^{-\bar{r}T}}} x_e \quad (98)$$

Consider the law of motion of  $\omega_{e,t} = \int_{T-t}^t f(t-s)\omega_{s,t}ds$ :

$$\begin{aligned} \frac{d\omega_{e,t}}{\omega_{e,t}} &= \int_{t-T}^t \frac{f(t-s)\omega_{s,t}}{\omega_{e,t}} \frac{d\omega_{s,t}}{\omega_{s,t}} ds + \left[ \frac{f(0)\omega_{t,t} - f(T)\omega_{t-T,t}}{\omega_{e,t}} - g \right] \\ &= \left[ r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \int_0^T \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}} \hat{\omega}_a da + \frac{\hat{h}_0}{x_e} - g \right] dt + \sigma_A dZ_t \end{aligned}$$

In a stationary equilibrium, the drift on  $\omega_{e,t}$  plus the growth rate of the population must equal the drift on the total wealth of financiers:

$$r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \int_0^T \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}} \hat{\omega}_a da + \frac{\hat{h}_0}{x_e} = r + \frac{(p^{ag})^2}{\gamma} - \bar{r}_f$$

Solving for  $p^{id}$ , we obtain

$$\frac{(p^{id})^2}{\gamma} = \int_0^T \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}} \hat{\omega}_a da - \bar{r}_f - \frac{\hat{h}_0}{x_e}$$

The average MPC of entrepreneurs is given by

$$\begin{aligned} \int_0^T \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}} \hat{\omega}_a da &= \frac{\bar{r}}{1 - \psi e^{-\bar{r}T}} \frac{\int_0^T e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)a} da}{1 - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T}} \frac{\hat{h}_0}{\frac{1-\psi}{1-\psi e^{-\bar{r}T}} x_e} \\ &= \frac{\bar{r}}{\bar{r} - \left(\bar{r}_f + \frac{(p^{id})^2}{\gamma}\right)} \frac{1 - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T}}{1 - \psi e^{-\bar{r}T} - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T} (1 - \psi)} \frac{\hat{h}_0}{x_e} \end{aligned}$$

Combining the previous two equations:

$$\bar{r}_f + \frac{(p^{id})^2}{\gamma} = \left[ \frac{\bar{r}}{\bar{r} - \left(\bar{r}_f + \frac{(p^{id})^2}{\gamma}\right)} \frac{1 - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T}}{1 - \psi e^{-\bar{r}T} \left(1 - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma}\right)T}\right)} - 1 \right] \frac{\hat{h}_0}{x_e}$$

rearranging

$$x_e = \left[ \frac{\frac{e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T} - 1}{\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}} - \psi e^{-\bar{r}T} \left( \frac{e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma}\right)T} - 1}{\bar{r}_f + \frac{(p^{id})^2}{\gamma}} \right)}{1 - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T} - \psi e^{-\bar{r}T} \left(1 - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma}\right)T}\right)} \right] \hat{h}_0 \quad (99)$$

We can plug the expression above into the pricing equation for idiosyncratic risk to obtain

$$p^{id} = \gamma \phi \sigma_{id} \frac{q k^{1-\alpha}}{\chi_h (1-\alpha)} \frac{h}{h_0} \frac{1 - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T} - \psi e^{-\bar{r}T} \left(1 - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma}\right)T}\right)}{\frac{e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T} - 1}{\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}} - \psi e^{-\bar{r}T} \left( \frac{e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma}\right)T} - 1}{\bar{r}_f + \frac{(p^{id})^2}{\gamma}} \right)} \quad (100)$$

The expression above can be rewritten as

$$\frac{\alpha k^{\alpha-1}}{q} = \frac{\gamma \phi \sigma_{id}}{p^{id}} \frac{\alpha}{\chi_h (1-\alpha)} \frac{h}{h_0} \frac{1 - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T} - \psi e^{-\bar{r}T} \left(1 - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma}\right)T}\right)}{\frac{e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T} - 1}{\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}} - \psi e^{-\bar{r}T} \left( \frac{e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma}\right)T} - 1}{\bar{r}_f + \frac{(p^{id})^2}{\gamma}} \right)} \quad (101)$$

solving for capital, we obtain

$$r + p^{ag}\sigma_A + p^{id}\phi\sigma_{id} - (g + \mu) + \frac{\Phi(g + \delta)}{q} = \frac{\gamma\phi\sigma_{id}}{p^{id}} \frac{\alpha}{\chi h(1 - \alpha)} \frac{h}{h_0} \frac{1 - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T} - \psi e^{-\bar{r}T} \left(1 - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma}\right)T}\right)}{e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T} - 1 - \psi e^{-\bar{r}T} \left(\frac{e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma}\right)T} - 1}{\bar{r}_f + \frac{(p^{id})^2}{\gamma}}\right)}} \quad (102)$$

$$r + p^{ag}\sigma_A + p^{id}\phi\sigma_{id} - (g + \mu) + \frac{\Phi(g + \delta)}{q} = \frac{\alpha k^{\alpha-1}}{q} \quad (103)$$

## C Numerical solution and computation

### C.1 Calibration

**Demographic parameters:** the parameters  $(g, T)$  will be calibrated to match the population growth in Thailand,  $g = 0.3\%$ , and  $T$  will be set to  $T = 55$ , capturing the period from 25 year old, the beginning of the life cycle for us, and 80 years old, the final date we have good data on entrepreneurs.

**Technological parameters:** First, consider the parameters  $(\mu_A, \sigma_A, \sigma_{id})$ . The drift  $\mu_A$  will be set to 0.3% to capture the average growth rate of wealth measured by Pawasutipaisit and Townsend (2010), and  $(\sigma_A, \sigma_{id})$  will be chosen to match the volatility of aggregate and idiosyncratic returns, respectively, as estimated by Samphantharak and Townsend (2018). Consider now the parameters  $(\bar{\Phi}_0, \bar{\Phi}_1, \alpha, \delta)$ . The depreciation rate is set at  $\delta = 0.09$  and the share of capital will be  $\alpha = 0.35$ . The adjustment cost parameters will be chosen to match a relative price of capital  $q = 1$  and an investment rate of  $ir = 0.25$ . Notice the investment rate is given by

$$ir = \frac{\Phi(g + \delta)k}{y} = \frac{\Phi(g + \delta)}{q} \frac{qk}{y} \quad (104)$$

Combining the expression above with condition (35), we can solve for the capital-output ratio

$$\frac{qk}{y} = \frac{\alpha - ir}{r + p^{ag}\sigma_A + p^{id}\phi\sigma_{id} - (g + \mu_A)} \quad (105)$$

where  $r$  is a calibration target and the aggregate and idiosyncratic risk premium,  $p^{ag}\sigma_A$  and  $p^{id}\phi\sigma_{id}$ , can be measured directly in the data.

Given a target for the capital-output ratio, we obtain an equation involving  $\bar{\Phi}_0$  and  $\bar{\Phi}_1$ . A target for  $q$  will give us another equation involving the same parameters. Hence, the adjustment cost parameters solve the system

$$\begin{bmatrix} g + \delta & 0.5(g + \delta)^2 \\ 1 & g + \delta \end{bmatrix} \begin{bmatrix} \bar{\Phi}_0 \\ \bar{\Phi}_1 \end{bmatrix} = \begin{bmatrix} \frac{ir * q}{qk/y} \\ q \end{bmatrix} \quad (106)$$

**Financier's discount rate:** the only parameter for the financier we need to calibrate is the discount

rate  $\rho_f$ . We will use the expression for the interest rate to calibrate  $\rho_f$ :

$$r = \rho_f + (g + \mu_A) - p^{ag}\sigma_A \Rightarrow \rho_f = r + p^{ag}\sigma_A - (g + \mu_A) \quad (107)$$

**Entrepreneur's preferences:** we need to calibrate the parameters  $(\rho, \psi, \gamma)$ . The risk aversion parameter  $\gamma$  can be obtained from the pricing equation for aggregate risk:

$$\gamma = \frac{p^{ag}}{\sigma_A} \quad (108)$$

where  $p^{ag}$ , the Sharpe ratio of aggregate risk, and  $\sigma_A$  can be measured directly in the data.

Given  $\bar{r}$ , the discount rate  $\rho$  is obtained by the condition

$$\rho = \gamma\bar{r} - (\gamma - 1) \left[ r + \frac{(p^{ag})^2}{2\gamma} + \frac{(p^{id})^2}{2\gamma} \right] \quad (109)$$

where  $p^{id}$  can be recovered from the data given  $\phi$ .

The parameters  $(\bar{r}, \psi)$  can be obtained as follows. The consumption-wealth ratio can be written as

$$\frac{c_{i,t}}{n_{i,t}} = \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-(t-s_i))}} \left( 1 + \frac{h_{i,t}}{n_{i,t}} \right) \quad (110)$$

Aggregating over all entrepreneurs with  $s_i = s$ , we obtain

$$\frac{c_{s,t}}{\omega_{s,t}} = \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-(t-s))}} \quad (111)$$

where  $\omega_{s,t} = n_{s,t} \left( 1 + \frac{h_{s,t}}{n_{s,t}} \right)$ .

Taking the average across all age groups:

$$\begin{aligned} \bar{c\omega} &\equiv \frac{1}{T} \int_{t-T}^t \frac{c_{s,t}}{\omega_{s,t}} ds = \frac{1}{T} \int_0^T \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}} da \\ &= \bar{r} + \frac{1}{T} \log \left( \frac{1 - \psi e^{-\bar{r}T}}{1 - \psi} \right) \end{aligned}$$

Rearranging the expression above, we obtain

$$e^{-\bar{r}T} = \frac{e^{-\bar{c\omega}T}}{1 - \psi + \psi e^{-\bar{c\omega}T}} \quad (112)$$

Taking the ratio of the consumption-wealth ratio at the beginning and end of life, we obtain

$$\frac{\bar{c\omega}_0}{\bar{c\omega}_T} = \frac{1 - \psi}{1 - \psi e^{-\bar{r}T}} \quad (113)$$



Solving for  $\psi$  and plugging the expression for  $e^{-\bar{r}T}$ , we get

$$\psi = \frac{1 - \frac{\bar{c}\bar{\omega}_0}{\bar{c}\bar{\omega}_T}}{1 - e^{-\bar{c}\bar{\omega}T}} \quad (114)$$

The coefficient  $\bar{r}$  is given by

$$\bar{r} = \bar{c}\bar{\omega} + \frac{1}{T} \log \left( \frac{\bar{c}\bar{\omega}_0}{\bar{c}\bar{\omega}_T} \right) \quad (115)$$

**Human wealth parameters:** The total labor income of entrepreneurs is given by  $w_t \bar{l}_e$ . The labor income of an entrepreneur of age  $a$  satisfies

$$w_t \bar{l}_{t-a,t} = w_t \bar{l}_e \sum_{k=1}^K \Gamma_k e^{\varphi_k a} \quad (116)$$

where

$$\int_0^T \sum_{k=1}^K \Gamma_k e^{\varphi_k a} f(a) da = 1 \quad (117)$$

Given data on the labor income of entrepreneurs by age group, the parameters  $(\Gamma_k, \varphi_k)_{k=1}^K$  can be estimated by non-linear least squares.

**Moral hazard parameter:** The share of financial wealth held by entrepreneurs with age  $a$  is given by

$$n_a = \frac{e^{\left[ r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - \bar{r} \right] a} \frac{1 - \psi e^{-\bar{r}(T-a)}}{1 - \psi e^{-\bar{r}T}}}{1 - e^{\left( r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - \bar{r} \right) T} \frac{1 - \psi}{1 - \psi e^{-\bar{r}T}}} h_0 - h_a \quad (118)$$

The human-financial wealth ratio is then given by

$$\frac{h_a}{n_a} = \frac{h_a/h_0}{\frac{e^{\left[ r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - \bar{r} \right] a} \frac{1 - \psi e^{-\bar{r}(T-a)}}{1 - \psi e^{-\bar{r}T}}}{1 - e^{\left( r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - \bar{r} \right) T} \frac{1 - \psi}{1 - \psi e^{-\bar{r}T}}} - h_a/h_0} \quad (119)$$

where

$$\frac{h_a}{h_0} = e^{-ga} \frac{\sum_{k=1}^K \Gamma_k e^{\varphi_k a} \frac{1 - e^{-(r + p^{ag}\sigma_A - \mu_A - \varphi_k)(T-a)}}{r + p^{ag}\sigma_A - \mu_A - \varphi_k}}{\sum_{k=1}^K \Gamma_k \frac{1 - e^{-(r + p^{ag}\sigma_A - \mu_A - \varphi_k)T}}{r + p^{ag}\sigma_A - \mu_A - \varphi_k}} \quad (120)$$

Except for  $p^{id}$ , all the other parameters that determine  $h_a/n_a$  have already been determined. We will choose  $p^{id}$  to match the average human-financial wealth ratio. Given  $p^{id}$ , we can obtain  $\phi$  using the condition

$$\phi = s^{id}/p^{id} \quad (121)$$

where  $s^{id}$  is the idiosyncratic Sharpe ratio, i.e., the idiosyncratic risk premium divided by the idiosyncratic volatility.

**Labor supply of entrepreneurs:** The price of idiosyncratic risk satisfies the condition

$$p^{id} = \gamma\phi\sigma_{id} \frac{qk}{n_e(1 + \frac{h_e}{n_e})} \quad (122)$$

where

$$\frac{qk}{n_e(1 + \frac{h_e}{n_e})} = \frac{qk}{(1-\alpha)y} \frac{(1-\alpha)k^\alpha}{h_0} \frac{1 - e^{\left(\rho_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T} - \psi e^{-\bar{r}T} \left(1 - e^{\left(\rho_f + \frac{(p^{id})^2}{\gamma}\right)T}\right)}{e^{\left(\rho_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T} - 1 - \psi e^{-\bar{r}T} \left(\frac{e^{\left(\rho_f + \frac{(p^{id})^2}{\gamma}\right)T} - 1}{\rho_f + \frac{(p^{id})^2}{\gamma}}\right)} \quad (123)$$

The capital-output ratio will be calibrated by the technological parameters. The parameters in  $\chi_h$  and  $h_0/h$  were already determined as well as  $\bar{r}$ . The only remaining term in the expression above is  $p^{id}$ . Solving for  $p^{id}$ , we can determine  $\phi$  as follows

$$\phi^2 = \frac{p^{id}\phi}{\gamma\sigma_{id}} \left[ \frac{qk}{n_e(1 + \frac{h_e}{n_e})} \right]^{-1} \quad (124)$$

$$r = \rho_f + (g + \mu_A) - \gamma\sigma_A^2 \quad (125)$$

The term  $\bar{r}_f$  is given by

$$\bar{r}_f = r + \gamma\sigma_A^2 - (g + \mu) = \rho_f \quad (126)$$

## C.2 Stationary equilibrium

In order to solve for a stationary equilibrium, we need to solve for  $(k, p^{id}, x_e)$ , using the conditions

$$\begin{aligned} r + p^{ag}\sigma_A + p^{id}\phi\sigma_{id} &= \frac{\alpha k^{\alpha-1} - \Phi(g + \delta)}{q} + g + \mu_A \\ p^{id} &= \gamma\phi\sigma_{id} \frac{qk}{qk + h} \frac{1}{x_e} \\ g + \mu_A &= r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \int_0^T \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}} \omega_a da + \frac{\hat{h}_0}{x_e} \end{aligned}$$

where  $r = \rho_f + (g + \mu_A) - \gamma\sigma_A^2$ ,  $p^{ag} = \gamma\sigma_A$ ,  $h = \chi_h(1 - \alpha)k^\alpha$  and

$$\chi_h = \sum_{k=1}^K \Gamma_k \int_0^T \frac{g e^{-ga}}{1 - e^{-gT}} e^{\varphi_k a} \frac{1 - e^{-(\rho_f + g - \varphi_k)(T-a)}}{\rho_f + g - \varphi_k} \bar{l}_e da + \frac{\bar{l}_f}{\rho_f} \quad (127)$$

Combining the first two equations, we can solve for  $x_e$  as a function of  $k$ :

$$x_e = \gamma(\phi\sigma_{id})^2 \frac{qk}{qk + h} \left[ \frac{\alpha k^{\alpha-1} - \Phi(g + \delta)}{q} - \rho_f \right]^{-1}$$

$$\begin{aligned}
\int_0^T \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}} \hat{\omega}_a da &= \frac{\bar{r}}{1 - \psi e^{-\bar{r}T}} \frac{\int_0^T e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)a} da}{1 - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T}} \frac{\hat{h}_0}{\frac{1-\psi}{1-\psi e^{-\bar{r}T}} x_e} \\
&= \frac{\bar{r}}{\bar{r} - \left(\bar{r}_f + \frac{(p^{id})^2}{\gamma}\right)} \frac{1 - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T}}{1 - \psi e^{-\bar{r}T} - e^{\left(\bar{r}_f + \frac{(p^{id})^2}{\gamma} - \bar{r}\right)T} (1 - \psi)} \frac{\hat{h}_0}{x_e}
\end{aligned}$$

## D Transitional dynamics

**Entrepreneurs:** The HJB for the entrepreneur is given by

$$\rho V_{i,t}(n_{i,t}; x_{i,t}) = \max_{c_{i,t}, k_{i,t}, l_{i,t}, \theta_{i,t}^{ag}, \theta_{i,t}^{id}} \left\{ \frac{c_{i,t}^{1-\gamma}}{1-\gamma} + \frac{\mathbb{E}_t [dV_{i,t}]}{dt} \right\}$$

subject to (7) and the boundary condition  $V_{i,s_i+T} = (1-\psi)\gamma V^* \frac{n_{i,s_i+T}^{1-\gamma}}{1-\gamma}$ , taking as given the deterministic path of interest rate,  $r_t$ , normalized wages,  $\tilde{w}_t = w_t/A_t$ , and normalized relative price of capital,  $\tilde{q}_t = q_t/A_t$ .

As before, we will guess-and-verify that  $V_{i,t}(n_{i,t}; h_{i,t}) = V_{s_i,t}^* \frac{(n_{i,t} + h_{i,t})^{1-\gamma}}{1-\gamma}$ , where  $V_{s_i,t}^*$  is a deterministic function of  $t$ , solves the PDE above. The HJB equation can be written more explicitly as

$$\begin{aligned}
\frac{\rho}{1-\gamma} = \max_{c_{i,t}, k_{i,t}, l_{i,t}, \theta_{i,t}^{ag}, \theta_{i,t}^{id}} &\left\{ \frac{1}{1-\gamma} \frac{1}{V_{s_i,t}^*} \left( \frac{c_{i,t}}{\omega_{i,t}} \right)^{1-\gamma} + \frac{1}{1-\gamma} \frac{\dot{V}_{s_i,t}^*}{V_{s_i,t}^*} + r_t + \frac{q_t k_{i,t}}{\omega_{i,t}} (\mu_{i,t}^R - r_t - p_t^{ag} \theta_{i,t}^{ag}) + \frac{h_{i,t}}{\omega_{i,t}} \sigma_A p_t^{ag} \right. \\
&\left. - \frac{c_{i,t}}{\omega_{i,t}} - \frac{\gamma}{2} \left[ \left( \frac{q_t k_{i,t}}{\omega_{i,t}} (\sigma_A - \theta_{i,t}^{ag}) + \frac{h_{i,t}}{\omega_{i,t}} \sigma_A \right)^2 + \left( \frac{q_t k_{i,t}}{\omega_{i,t}} (\sigma_{id} - \theta_{i,t}^{id}) \right)^2 \right] \right\}
\end{aligned}$$

where

$$\mu_{i,t}^R = \frac{k_{i,t}^\alpha l_{i,t}^{1-\alpha} - \tilde{w}_t l_{i,t} - \Phi(l_{i,t}) k_{i,t}}{\tilde{q}_t k_{i,t}} + \mu_A + \frac{\dot{\tilde{q}}_t}{\tilde{q}_t} + l_{i,t} - \delta \quad (128)$$

The first-order conditions with respect to  $l_{i,t}$  and  $k_{i,t}$  are given by

$$\begin{aligned}
\tilde{w}_t &= (1-\alpha) \left( \frac{k_{i,t}}{l_{i,t}} \right)^\alpha = (1-\alpha) k_t^\alpha \\
\tilde{q}_t &= \bar{\Phi}_0 + \bar{\Phi} l_{i,t}
\end{aligned}$$

The expected return on the project will then be equalized across entrepreneurs:

$$\mu_{i,t}^R = \frac{k_t^\alpha - \tilde{w}_t l_{i,t} - \Phi(l_{i,t})}{\tilde{q}_t} + \mu_A + \frac{\dot{\tilde{q}}_t}{\tilde{q}_t} + l_{i,t} - \delta \quad (129)$$

Maximizing over  $(c_{i,t}, k_{i,t}, \theta_{i,t}^{ag}, \theta_{i,t}^{id})$  gives the conditions

$$\begin{aligned} p_t^{id} &= \frac{\mu_t^R - r_t - p_t^{ag} \sigma_A}{\phi \sigma_{id}} \\ \frac{q_t k_{i,t}}{\omega_{i,t}} &= \frac{p_t^{id}}{\gamma \phi \sigma_{id}} \\ \frac{c_{i,t}}{\omega_{i,t}} &= (V_{s_i,t}^*)^{-\frac{1}{\gamma}} \\ p_t^{ag} &= \gamma \left( \frac{q_t k_{i,t}}{\omega_{i,t}} (\sigma_A - \theta_{i,t}^{ag}) + \frac{h_{i,t}}{\omega_{i,t}} \sigma_A \right) \end{aligned}$$

Let  $z_{s_i,t} \equiv (V_{s_i,t}^*)^{\frac{1}{\gamma}}$  and  $\bar{r}_t = \frac{1}{\gamma} \rho + \left(1 - \frac{1}{\gamma}\right) \left[ r_t + \frac{(p_t^{id})^2 + (p_t^{ag})^2}{2\gamma} \right]$ , then plugging the expressions above into the HJB equation gives the following differential equation

$$e^{-\int_{s_i}^t \bar{r}_z dz} \dot{z}_{s_i,t} - \bar{r}_t e^{-\int_{s_i}^t \bar{r}_z dz} z_{s_i,t} = -e^{-\int_{s_i}^t \bar{r}_z dz} \quad (130)$$

Solving the differential equation:

$$e^{-\int_{s_i}^{s_i+T} \bar{r}_z dz} (1 - \psi) (V^*)^{\frac{1}{\gamma}} - e^{-\int_{s_i}^t \bar{r}_z dz} (V_{s_i,t}^*)^{\frac{1}{\gamma}} = - \int_t^{s_i+T} e^{-\int_{s_i}^{t'} \bar{r}_z dz} dt' \quad (131)$$

rearranging

$$(V_{s_i,t}^*)^{\frac{1}{\gamma}} = \int_t^{s_i+T} e^{-\int_{s_i}^{t'} \bar{r}_z dz} dt' + e^{-\int_{s_i}^{s_i+T} \bar{r}_z dz} (V^*)^{\frac{1}{\gamma}} \quad (132)$$

**Financiers:** The optimal consumption and portfolio decisions for entrepreneurs are still given by the conditions:

$$\begin{aligned} c_{f,t} &= \rho_f (n_{f,t} + h_{f,t}) \\ \theta_{f,t}^{ag} &= (n_{f,t} + h_{f,t}) \frac{p_t^{ag}}{\gamma} - h_{f,t} \sigma_A \end{aligned}$$

**Price of aggregate insurance:** The market clearing for aggregate insurance implies

$$\int_{t-T}^t [q_t k_{s,t} + h_{s,t}] ds \sigma_A - \int_{t-T}^t [n_{s,t} + h_{s,t}] ds \frac{p_t^{ag}}{\gamma} = (n_{f,t} + h_{f,t}) \frac{p_t^{ag}}{\gamma} - h_{f,t} \sigma_A \quad (133)$$

Solving the equation above gives  $p_t^{ag} = \gamma \sigma_A$ . Hence, after a change in  $\gamma$  or  $\sigma_A$  the price of aggregate risk jumps immediately to its new long-run level.

**Price of idiosyncratic insurance:** From the optimality condition for capital, we obtain the idiosyncratic risk premium:

$$p_t^{id} = \tilde{q}_t \tilde{k}_t \frac{\gamma \phi \sigma_{id}}{z_{e,t}} \quad (134)$$

where  $z_{e,t} = x_{e,t} (\tilde{q}_t \tilde{k}_t + \tilde{h}_t)$  is the total wealth of entrepreneurs normalized by  $A_t l_t$ .

The expected return on the business satisfy the condition

$$r_t + \gamma\sigma_A^2 + p_t^{id}\phi\sigma_{id} = \frac{a\tilde{k}_t^{\alpha-1} - \Phi(\iota_t)}{\tilde{q}_t} + \mu_A + \frac{\dot{\tilde{q}}_t}{\tilde{q}_t} + \iota_t - \delta \quad (135)$$

**Goods market:** The market clearing condition for goods is given by

$$\int_{t-T}^t f(t-s) [c_{s,t} + \Phi(\iota_{s,t})A_t k_{s,t}] ds + c_{f,t} = \int_{t-T}^t f(t-s)y_{s,t} ds$$

Using the expression for the consumption-wealth ratio, we get

$$\int_{t-T}^t [\zeta_{s,t}(n_{s,t} + h_{s,t})] f(t-s) ds + \rho_f(n_{s,t} + h_{s,t}) = \int_{t-T}^t f(t-s) \left[ k_{s,t}^\alpha l_{s,t}^{1-\alpha} - \Phi(\iota_{s,t})A_t k_{s,t} \right] ds$$

where  $\zeta_{s,t} \equiv (V_{s,t}^*)^{-\frac{1}{\gamma}}$ .

Dividing by the total wealth per capita gives

$$x_{e,t}\zeta_{e,t} + (1 - x_{e,t})\rho_f = \frac{\tilde{k}_t^\alpha - \Phi(\iota_t)\tilde{k}_t}{\tilde{q}_t\tilde{k}_t + \tilde{h}_t} \quad (136)$$

where  $x_{e,t} = \int_{t-T}^t \frac{n_{s,t} + h_{s,t}}{q_t k_t + h_t} f(t-s) ds$  and  $\tilde{k}_t$  denotes the capital-labor ratio.

**Law of motion of  $\tilde{\omega}_{s,t}$ :** Define  $\tilde{\omega}_{s,t} = \omega_{s,t} f(t-s) / \omega_{e,t}$ , then the law of motion of  $\tilde{\omega}_{s,t}$  is given by

$$\frac{\dot{\tilde{\omega}}_{s,t}}{\tilde{\omega}_{s,t}} = \zeta_{e,t} - \zeta_{s,t} - f(0) \frac{h_{t,t}}{\omega_{e,t}} \quad (137)$$

**Law of motion of  $x_{e,t}$ :** The share of total wealth held be entrepreneurs is given by

$$x_{e,t} = \frac{1 - e^{-gT}}{g} e^{gt} \frac{\omega_{e,t}}{q_t k_t + h_t} \quad (138)$$

The law of motion of the numerator is given by

$$\frac{d(e^{gt}\omega_{e,t})}{e^{gt}\omega_{e,t}} = \left[ r_t + \gamma\sigma_A^2 + \frac{(p_t^{id})^2}{\gamma} - \zeta_{e,t} + \frac{\hat{h}_{0,t}}{x_{e,t}} \right] dt + \sigma_A dZ_t$$

where  $\hat{h}_{0,t} \equiv e^{gt} h_{t,t} / (q_t k_t + h_t)$

The law of motion of the denominator is given by

$$\begin{aligned} \frac{d(q_t k_t + h_t)}{q_t k_t + h_t} &= \frac{1}{q_t k_t + h_t} \left[ \left( \mu_A + \frac{\dot{\tilde{q}}_t}{\tilde{q}_t} + \iota_t - \delta \right) q_t k_t + (r_t + \gamma\sigma_A^2) h_t - w_t l_t \right] dt + \sigma_A dZ_t \\ &= \left[ r_t + \gamma\sigma_A + p_t^{id}\phi\sigma_{id} \frac{\tilde{q}_t \tilde{k}_t}{\tilde{q}_t \tilde{k}_t + \tilde{h}_t} - \frac{\tilde{k}_t^\alpha - \Phi(\iota_t)}{\tilde{q}_t \tilde{k}_t + \tilde{h}_t} \right] dt + \sigma_A dZ_t \end{aligned}$$

using the condition (135).

Combining these two expressions, we can derive the law of motion of  $x_{e,t}$ :

$$\begin{aligned} \frac{dx_{e,t}}{x_{e,t}} &= \frac{d(e^{st}\omega_{e,t})}{e^{st}\omega_{e,t}} - \frac{d(q_t k_t + h_t)}{q_t k_t + h_t} + \left( \frac{d(q_t k_t + h_t)}{q_t k_t + h_t} \right)^2 - \frac{d(q_t k_t + h_t)}{q_t k_t + h_t} \frac{d(e^{st}\omega_{e,t})}{e^{st}\omega_{e,t}} \\ &= \left[ \frac{(p_t^{id})^2}{\gamma} - \zeta_{e,t} + \frac{\hat{h}_{0,t}}{x_{e,t}} - p_t^{id} \phi \sigma_{id} \frac{\tilde{q}_t \tilde{k}_t}{\tilde{q}_t \tilde{k}_t + \tilde{h}_t} + \frac{\tilde{k}_t^\alpha - \Phi(t_t)}{\tilde{q}_t \tilde{k}_t + \tilde{h}_t} \right] dt \end{aligned}$$

Using (134), we obtain

$$\dot{x}_{e,t} = \gamma(\phi\sigma_{id})^2 \left( \frac{\tilde{q}_t \tilde{k}_t}{\tilde{q}_t \tilde{k}_t + \tilde{h}_t} \right) \frac{1 - x_{e,t}}{x_{e,t}} + \hat{h}_{0,t} + x_{e,t}(1 - x_{e,t})(\rho_f - \zeta_{e,t}) \quad (139)$$

Finally, notice that the wealth of entrepreneurs is divided between capital and bonds, so  $x_{e,0}(q_0 k_0 + h_0) = q_0 k_0 + b_0$ . Hence, the initial share of wealth of entrepreneurs satisfy  $x_{e,0} = \frac{q_0 k_0 + b_0}{q_0 k_0 + h_0}$ , where  $b_0$  is given.

**Human wealth:** The human wealth of financiers is given by

$$\begin{aligned} \tilde{h}_{f,t} &= \int_t^\infty e^{-\int_t^z (r_s + \gamma\sigma_A^2 - g - \mu_A) ds} (1 - \alpha) \tilde{k}_z \bar{l}_f dz \\ &= \int_t^{t_{max}} e^{-\int_t^z (r_s + \gamma\sigma_A^2 - g - \mu_A) ds} (1 - \alpha) \tilde{k}_z \bar{l}_f dz + e^{-\int_t^{t_{max}} (r_s + \gamma\sigma_A^2 - g - \mu_A) ds} \frac{(1 - \alpha) \tilde{k}_{t_{max}}^\alpha}{r_{t_{max}} + \gamma\sigma_A^2 - (g + \mu_A)} \bar{l}_f \end{aligned}$$

where  $\tilde{h}_{f,t} \equiv h_{f,t} / (A_t e^{st} \frac{1 - e^{-sT}}{g})$  and using the fact

$$\mathbb{E}_t \left[ w_z \bar{l}_{f,z} \right] = \mathbb{E}_t \left[ \tilde{w}_z A_z e^{g(z-t)} \frac{1 - e^{-sT}}{g} \bar{l}_f \right] = (1 - \alpha) \tilde{k}_z^\alpha e^{(\mu_A + g)(z-t)} A_t e^{st} \frac{1 - e^{-sT}}{g} \bar{l}_f \quad (140)$$

The human-wealth of an entrepreneur born in period  $s$  is given by

$$\tilde{h}_{t-a,t} = \sum_{k=1}^K \Gamma_k e^{\varphi_k a} \int_0^{T-a} e^{-\int_t^{t+z} (r_s + \gamma\sigma_A^2 - \mu_A - \varphi_k) ds} (1 - \alpha) \tilde{k}_{t+z}^\alpha \bar{l}_e dz \quad (141)$$

where  $\tilde{h}_{s,t} \equiv h_{s,t} A_t$ .

The sum of human wealth for entrepreneurs and financiers is given by

$$\tilde{h}_t = \int_{t-T}^t \hat{h}_{s,t} f(t-s) ds + \hat{h}_{f,t} \quad (142)$$

where  $\tilde{h}_t \equiv h_t / (A_t e^{st} \frac{1 - e^{-sT}}{g})$ .

The average human wealth of entrepreneurs can be written as

$$\begin{aligned}
\int_{t-T}^t \hat{h}_{s,t} f(t-s) ds &= (1-\alpha) f(0) \bar{l}_e \sum_{k=1}^K \Gamma_k \int_0^T \int_0^{T-a} e^{(\varphi_k-g)a} e^{-\int_t^{t+z} (r_s + \gamma \sigma_A^2 - \mu_A - \varphi_k) ds} \tilde{k}_{t+z}^\alpha dz da \\
&= (1-\alpha) f(0) \bar{l}_e \sum_{k=1}^K \Gamma_k \int_0^T \int_0^{T-z} e^{(\varphi_k-g)a} e^{-\int_t^{t+z} (r_s + \gamma \sigma_A^2 - \mu_A - \varphi_k) ds} \tilde{k}_{t+z}^\alpha da dz \\
&= (1-\alpha) f(0) \bar{l}_e \sum_{k=1}^K \Gamma_k \int_0^T \frac{e^{(\varphi_k-g)(T-z)-1}}{\varphi_k - g} e^{-\int_t^{t+z} (r_s + \gamma \sigma_A^2 - \mu_A - \varphi_k) ds} \tilde{k}_{t+z}^\alpha dz \\
&= (1-\alpha) f(0) \bar{l}_e \sum_{k=1}^K \frac{\Gamma_k}{\varphi_k - g} \int_0^T \left[ e^{(\varphi_k-g)T} e^{-\int_t^{t+z} (r_s + \gamma \sigma_A^2 - \mu_A - g) ds} - e^{-\int_t^{t+z} (r_s + \gamma \sigma_A^2 - \mu_A - \varphi_k) ds} \right] \tilde{k}_{t+z}^\alpha dz
\end{aligned}$$

If  $t > t_{max} - T$ , then we can write

$$\begin{aligned}
\int_{t-T}^t \hat{h}_{s,t} f(t-s) ds &= (1-\alpha) f(0) \bar{l}_e \sum_{k=1}^K \frac{\Gamma_k}{\varphi_k - g} \int_0^{t_{max}} \left[ e^{(\varphi_k-g)T} e^{-\int_t^{t+z} (r_s + \gamma \sigma_A^2 - \mu_A - g) ds} - e^{-\int_t^{t+z} (r_s + \gamma \sigma_A^2 - \mu_A - \varphi_k) ds} \right] \tilde{k}_{t+z}^\alpha dz \\
&\quad + (1-\alpha) f(0) \bar{l}_e \sum_{k=1}^K \frac{\Gamma_k}{\varphi_k - g} \int_{t_{max}}^T \left[ e^{(\varphi_k-g)T} e^{-\int_t^{t+z} (r_s + \gamma \sigma_A^2 - \mu_A - g) ds} - e^{-\int_t^{t+z} (r_s + \gamma \sigma_A^2 - \mu_A - \varphi_k) ds} \right] \tilde{k}_{t+z}^\alpha dz
\end{aligned}$$

The term  $\hat{h}_{0,t} \equiv f(0) \tilde{h}_{t,t} / (\tilde{q}_t \tilde{k}_t + \tilde{h}_t)$  is given by

$$\hat{h}_{0,t} = \frac{g}{1 - e^{-gT}} \sum_{k=1}^K \Gamma_k \int_t^{t+T} e^{-\int_t^z (r_s + \gamma \sigma_A^2 - \mu_A - \varphi_k) ds} \frac{(1-\alpha) \tilde{k}_z^\alpha \bar{l}_e}{\tilde{q}_t \tilde{k}_t + \tilde{h}_t} dz \quad (143)$$

**Discretized equations:** For the numerical solution of the equilibrium, we will discretize the differential equations to obtain difference equations in the grid  $t \in \{0, \Delta t, \dots, t_{max}\}$ . The equations describing the equilibrium consists of the following conditions

- *Prices:*

$$\begin{aligned}
\tilde{q}_t &= \frac{1}{\tilde{k}_t} \left[ \frac{\tilde{k}_t^\alpha - \Phi(t_t) \tilde{k}_t}{x_{e,t} \zeta_{e,t} + (1 - x_{e,t}) \rho_f} - \tilde{h}_t \right] \\
p_t^{id} &= \frac{\tilde{q}_t \tilde{k}_t}{\tilde{q}_t \tilde{k}_t + \tilde{h}_t} \frac{\gamma \phi \sigma_{id}}{x_{e,t}} \\
r_t &= \frac{\alpha \tilde{k}_t^{\alpha-1} - \Phi(t_t)}{\tilde{q}_t} + \mu_A + \frac{\tilde{q}_{t+\Delta t} - \tilde{q}_t}{\Delta t} \frac{1}{\tilde{q}_t} + \frac{\tilde{k}_{t+\Delta t} - \tilde{k}_t}{\Delta t} \frac{1}{\tilde{k}_t} + g - \gamma \sigma_A^2 - p_t^{id} \phi \sigma_{id}
\end{aligned}$$

- *Consumption-wealth ratio by age:*

$$\zeta_{t-a,t}^{-1} = \sum_{k=0}^{n_a} \exp \left( - \sum_{i=1}^k \bar{r}_{t+i\Delta} \Delta \right) \Delta - \frac{1 + \exp \left( - \sum_{i=1}^{n_a} \bar{r}_{t+i\Delta} \Delta \right)}{2} \Delta + \exp \left( - \sum_{i=1}^{n_a} \bar{r}_{t+i\Delta} \Delta \right) (V^*)^{\frac{1}{\gamma}}$$

where  $n_a \equiv (T - a) / \Delta t$  and  $\bar{r}_t = \frac{1}{\gamma} \rho + \left( 1 - \frac{1}{\gamma} \right) \left[ \frac{r_t + r_{t-\Delta}}{2} + \frac{(p_t^{id})^2}{2\gamma} + \frac{\gamma \sigma_A^2}{2} \right]$ .

$$\begin{aligned}\zeta_{t+\Delta-a,t+\Delta}^{-1} &= \sum_{k=0}^{n_a} \exp\left(-\sum_{i=1}^k \bar{r}_{t+(i+1)\Delta}\Delta\right) \Delta - \frac{\Delta}{2} + \exp\left(-\sum_{i=1}^{n_a} \bar{r}_{t+(i+1)\Delta}\Delta\right) \left((V^*)^{\frac{1}{\gamma}} - \frac{\Delta}{2}\right) \\ &= \sum_{k=1}^{n_a+1} \exp\left(-\sum_{i=2}^k \bar{r}_{t+i\Delta}\Delta\right) \Delta - \frac{\Delta}{2} + \exp\left(-\sum_{i=2}^{n_a+1} \bar{r}_{t+i\Delta}\Delta\right) \left((V^*)^{\frac{1}{\gamma}} - \frac{\Delta}{2}\right)\end{aligned}$$

This implies the recursion

$$\zeta_{t-a,t}^{-1} = \frac{1 + e^{-\bar{r}_{t+\Delta}\Delta}}{2} \Delta + e^{-\bar{r}_{t+\Delta}\Delta} \zeta_{t+\Delta-(a+\Delta),t+\Delta}^{-1} \quad (144)$$

for  $a = 0, \Delta, \dots, T - \Delta$ . and

$$\zeta_{t-T,t}^{-1} = (1 - \psi) \frac{1}{\bar{r}_1} \quad (145)$$

For  $a = 0$ , we have

$$\begin{aligned}\zeta_{t,t}^{-1} &= \sum_{k=0}^{n_0} \exp\left(-\sum_{i=1}^k \bar{r}_{t+i\Delta}\Delta\right) \Delta - \frac{1 + \exp\left(-\sum_{i=1}^{n_0} \bar{r}_{t+i\Delta}\Delta\right)}{2} \Delta + \exp\left(-\sum_{i=1}^{n_0} \bar{r}_{t+i\Delta}\Delta\right) (V^*)^{\frac{1}{\gamma}} \\ \zeta_{t+\Delta-T,t+\Delta}^{-1} &= \sum_{k=1}^{n_0+1} \exp\left(-\sum_{i=2}^k \bar{r}_{t+i\Delta}\Delta\right) \Delta - \frac{1 + \exp\left(-\sum_{i=2}^{n_0+1} \bar{r}_{t+i\Delta}\Delta\right)}{2} \Delta + \exp\left(-\sum_{i=1}^{n_0+1} \bar{r}_{t+i\Delta}\Delta\right) (V^*)^{\frac{1}{\gamma}}\end{aligned}$$

Combining the two expressions we obtain

$$\zeta_{t,t}^{-1} = \frac{1 + e^{-\bar{r}_{t+\Delta}\Delta}}{2} \Delta + e^{-\bar{r}_{t+\Delta}\Delta} \zeta_{t+\Delta-a,t+\Delta}^{-1} + \left(e^{(-\sum_{i=1}^{n_0} \bar{r}_{t+i\Delta}\Delta)} - e^{(-\sum_{i=1}^{n_0+1} \bar{r}_{t+i\Delta}\Delta)}\right) \left((V^*)^{\frac{1}{\gamma}} - \frac{\Delta}{2}\right) \quad (146)$$

- *Wealth shares and average consumption-wealth ratio:*

$$\tilde{\omega}_{s,t+\Delta t} = \begin{cases} \left[1 + \left(\zeta_{e,t} - \zeta_{s,t} - \frac{\hat{h}_{0,t}}{x_{e,t}}\right) \Delta t\right] \tilde{\omega}_{s,t}, & \text{for } s \in \{t + \Delta t - T, t + 2\Delta t - T, \dots, t\} \\ \tilde{\omega}_{t+\Delta-T,t+\Delta t} + \frac{\hat{h}_{0,t+\Delta t}}{x_{e,t+\Delta t}}, & \text{for } s = t + \Delta t \end{cases}$$

where  $\tilde{\omega}_{s,0}$  for  $s \in \{-T, \Delta t - T, \dots, 0\}$  is given.

Adding the difference equation above for all  $s$ , we obtain

$$\sum_{k=0}^{n_T-1} [\tilde{\omega}_{t-k\Delta t,t+\Delta t} - \tilde{\omega}_{t-k\Delta t,t}] = \sum_{k=0}^{n_T-1} \left[ \zeta_{e,t} \tilde{\omega}_{t-k\Delta t,t} - \zeta_{t-k\Delta t,t} \tilde{\omega}_{t-k\Delta t,t} - \frac{\hat{h}_{0,t}}{x_{e,t}} \tilde{\omega}_{t-k\Delta t,t} \right] \Delta$$



Since  $\int_0^T \tilde{\omega}_{t-a,t} da = 1$  for all  $t$ , we must have

$$\sum_{k=0}^{n_T-1} \tilde{\omega}_{t-k\Delta,t} - \frac{\tilde{\omega}_{t,t} - \tilde{\omega}_{t-T,t}}{2} = \sum_{k=0}^{n_T-1} \tilde{\omega}_{t-k\Delta,t+\Delta} + \frac{\tilde{\omega}_{t+\Delta,t+\Delta} - \tilde{\omega}_{t+\Delta-T,t+\Delta}}{2} = \frac{1}{\Delta} \quad (147)$$

applying the trapezoidal rule to approximate the integral.

Similarly, the average consumption-wealth ratio for entrepreneurs is

$$\zeta_{e,t} = \sum_{k=0}^{n_0} \tilde{\omega}_{t-k\Delta,t} \zeta_{t-k\Delta,t} \Delta t - \frac{\tilde{\omega}_{t,t} \zeta_{t,t} + \tilde{\omega}_{t-T,t} \zeta_{t-T,t}}{2} \quad (148)$$

Hence, we can write

$$\begin{aligned} \frac{\tilde{\omega}_{t+\Delta,t+\Delta} - \tilde{\omega}_{t+\Delta-T,t+\Delta}}{2} &= - \sum_{k=0}^{n_T-1} [\tilde{\omega}_{t-k\Delta,t+\Delta} - \tilde{\omega}_{t-k\Delta,t}] - \frac{1}{2} \frac{\hat{h}_{0,t}}{x_{e,t}} \\ &= \sum_{k=0}^{n_T-1} \left[ \zeta_{t-k\Delta,t} \tilde{\omega}_{t-k\Delta,t} + \frac{\hat{h}_{0,t}}{x_{e,t}} \tilde{\omega}_{t-k\Delta,t} - \zeta_{e,t} \tilde{\omega}_{t-k\Delta,t} \right] \Delta - \frac{1}{2} \frac{\hat{h}_{0,t}}{x_{e,t}} \\ &= \zeta_{e,t} + \frac{\tilde{\omega}_{t,t} \zeta_{t,t} - \tilde{\omega}_{t-T,t} \zeta_{t-T,t}}{2} \Delta + \left( \frac{\hat{h}_{0,t}}{x_{e,t}} - \zeta_{e,t} \right) \left( 1 + \frac{\tilde{\omega}_{t,t} - \tilde{\omega}_{t-T,t}}{2} \Delta \right) - \frac{1}{2} \frac{\hat{h}_{0,t}}{x_{e,t}} \\ &= \frac{1}{2} \frac{\hat{h}_{0,t}}{x_{e,t}} + \left[ \tilde{\omega}_{t,t} \zeta_{t,t} - \tilde{\omega}_{t-T,t} \zeta_{t-T,t} + \left( \frac{\hat{h}_{0,t}}{x_{e,t}} - \zeta_{e,t} \right) (\tilde{\omega}_{t,t} - \tilde{\omega}_{t-T,t}) \right] \frac{\Delta}{2} \end{aligned}$$

This allow us to solve for  $\tilde{\omega}_{t+\Delta,t+\Delta}$ :

$$\tilde{\omega}_{t+\Delta,t+\Delta} = \tilde{\omega}_{t+\Delta-T,t+\Delta} + \frac{\hat{h}_{0,t}}{x_{e,t}} + \left[ \tilde{\omega}_{t,t} \zeta_{t,t} - \tilde{\omega}_{t-T,t} \zeta_{t-T,t} + \left( \frac{\hat{h}_{0,t}}{x_{e,t}} - \zeta_{e,t} \right) (\tilde{\omega}_{t,t} - \tilde{\omega}_{t-T,t}) \right] \Delta \quad (149)$$

Notice that as  $\Delta \rightarrow 0$ , we recover the condition  $\tilde{\omega}_{t,t} = \tilde{\omega}_{t-T,t} + \frac{\hat{h}_{0,t}}{x_{e,t}}$ .

$$\begin{aligned} \sum_{k=0}^{n_T} \tilde{\omega}_{t+\Delta-k\Delta,t} - \frac{\tilde{\omega}_{t+\Delta,t+\Delta} + \tilde{\omega}_{t+\Delta-T,t+\Delta}}{2} &= \sum_{k=1}^{n_T} \tilde{\omega}_{t+\Delta-k\Delta,t} - \frac{1}{2} \left( \frac{\hat{h}_{0,t}}{x_{e,t}} + \left[ \tilde{\omega}_{t,t} \zeta_{t,t} - \tilde{\omega}_{t-T,t} \zeta_{t-T,t} + \left( \frac{\hat{h}_{0,t}}{x_{e,t}} - \zeta_{e,t} \right) (\tilde{\omega}_{t,t} - \tilde{\omega}_{t-T,t}) \right] \Delta \right) \\ &= \sum_{k=1}^{n_T} \tilde{\omega}_{t-k\Delta,t} - \frac{1}{2} \left( \frac{\hat{h}_{0,t}}{x_{e,t}} + \left[ \tilde{\omega}_{t,t} \zeta_{t,t} - \tilde{\omega}_{t-T,t} \zeta_{t-T,t} + \left( \frac{\hat{h}_{0,t}}{x_{e,t}} - \zeta_{e,t} \right) (\tilde{\omega}_{t,t} - \tilde{\omega}_{t-T,t}) \right] \Delta \right) \end{aligned}$$

$$\begin{aligned} \left[ \sum_{k=0}^{n_T} \tilde{\omega}_{t+\Delta-k\Delta,t+\Delta} - \frac{\tilde{\omega}_{t+\Delta,t+\Delta} + \tilde{\omega}_{t+\Delta-T,t+\Delta}}{2} \right] \Delta &= 1 \\ \left[ \sum_{k=1}^{n_T} \tilde{\omega}_{t+\Delta-k\Delta,t+\Delta} + \frac{\tilde{\omega}_{t+\Delta,t+\Delta} - \tilde{\omega}_{t+\Delta-T,t+\Delta}}{2} \right] \Delta &= 1 \end{aligned}$$

which imply

$$\begin{aligned}
\tilde{\omega}_{t+\Delta,t+\Delta} &= \tilde{\omega}_{t+\Delta-T,t+\Delta} + 2 \left[ 1 - \sum_{k=1}^{n_T} \tilde{\omega}_{t+\Delta-k\Delta,t+\Delta} \Delta \right] \frac{1}{\Delta} \\
&= \tilde{\omega}_{t+\Delta-T,t+\Delta} + 2 \left[ 1 - \sum_{k=1}^{n_T} \tilde{\omega}_{t-k\Delta,t} \Delta + \sum_{k=1}^{n_T} \left( \zeta_{t-k\Delta,t} + \frac{\hat{h}_{0,t}}{x_{e,t}} - \zeta_{e,t} \right) \tilde{\omega}_{t-k\Delta,t} \Delta \right] \frac{1}{\Delta} \\
&= \tilde{\omega}_{t+\Delta-T,t+\Delta} + \tilde{\omega}_{t,t} - \tilde{\omega}_{t-T,t} + 2 \sum_{k=1}^{n_T} \left( \zeta_{t-k\Delta,t} + \frac{\hat{h}_{0,t}}{x_{e,t}} - \zeta_{e,t} \right) \tilde{\omega}_{t-k\Delta,t}
\end{aligned}$$

• *Human wealth: Financiers*

$$\tilde{h}_{f,t} = (1-\alpha)\bar{l}_f \left[ \sum_{k=0}^{n_{t,max}} \exp\left(-\sum_{i=1}^k r_{t+i\Delta}^h \Delta\right) \tilde{k}_{t+k\Delta}^\alpha - \frac{\tilde{k}_t^\alpha + e^{-\sum_{i=1}^{n_{t,max}} r_{t+i\Delta}^h \Delta} \tilde{k}_{t_{max}}^\alpha}{2} \right] \Delta + \exp\left(-\sum_{i=1}^{n_{t,max}} r_{t+i\Delta}^h \Delta\right) \tilde{h}_{f,t_{max}}$$

where  $n_{t,max} \equiv (t_{max} - t)/\Delta$ ,  $r_t^h \equiv \frac{r_t + r_{t-\Delta}}{2} + \gamma\sigma_A^2 - (g + \mu_A)$ .

Updating the equation above one-period, we get

$$\tilde{h}_{f,t+\Delta} = (1-\alpha)\bar{l}_f \left[ \sum_{k=1}^{n_{t,max}} \exp\left(-\sum_{i=2}^k r_{t+i\Delta}^h \Delta\right) \tilde{k}_{t+k\Delta}^\alpha - \frac{\tilde{k}_{t+\Delta}^\alpha + e^{-\sum_{i=2}^{n_{t,max}} r_{t+i\Delta}^h \Delta} \tilde{k}_{t_{max}}^\alpha}{2} \right] \Delta + \exp\left(-\sum_{i=2}^{n_{t,max}} r_{t+i\Delta}^h \Delta\right) \tilde{h}_{f,t_{max}}$$

Human wealth of financiers is then given by

$$\tilde{h}_{f,t} = (1-\alpha)\bar{l}_f \left[ \frac{\tilde{k}_t^\alpha + e^{-r_{t+\Delta}^h \Delta} \tilde{k}_{t+\Delta}^\alpha}{2} \right] \Delta + e^{-r_{t+\Delta}^h \Delta} \tilde{h}_{f,t+\Delta} \quad (150)$$

Entrepreneurs of age  $a$ :

$$\tilde{h}_{t-a,t} = (1-\alpha)\bar{l}_e \sum_{k=1}^K \Gamma_k e^{\varphi_k a} \sum_{j=1}^{n_a} \exp\left(-\sum_{i=1}^j r_{t+i\Delta}^k \Delta t\right) \tilde{k}_{t+j\Delta t}^\alpha \Delta t \quad (151)$$

where  $r_t^k \equiv \frac{r_t + r_{t-\Delta t}}{2} + \gamma\sigma_A^2 - \mu_A - \varphi_k$ .

Average human wealth of entrepreneurs is given by

$$\tilde{h}_{e,t} \equiv (1-\alpha)f(0)\bar{l}_e \sum_{k=1}^K \frac{\Gamma_k}{\varphi_k - g} \left[ \sum_{j=0}^{n_T} \tilde{\delta}_{t,j\Delta}^k \tilde{k}_{t+j\Delta}^\alpha - \frac{(e^{(\varphi_k - g)T} - 1)\tilde{k}_t^\alpha + \tilde{\delta}_{t,T}^k \tilde{k}_{t+T}^\alpha}{2} \right] \Delta$$

where

$$\tilde{\delta}_{t,z}^k \equiv e^{(\varphi_k - g)T} e^{-\sum_{i=1}^{z/\Delta} r_{t+i\Delta}^k} - e^{-\sum_{i=1}^{z/\Delta} r_{t+i\Delta}^k} \quad (152)$$

We can write the expression above in recursive form, by defining the following objects:

$$\tilde{h}_{e,t}^4 = (1 - \alpha)f(0)\bar{l}_e \sum_{k=1}^K \frac{\Gamma_k}{\varphi_k - g} e^{(\varphi_k - g)T} \left[ \sum_{j=0}^{n_T} e^{-\sum_{i=1}^j r_{t+i\Delta}^h \Delta} \tilde{k}_{t+j\Delta}^\alpha - \frac{\tilde{k}_t^\alpha + e^{-\sum_{i=1}^{n_T} r_{t+i\Delta}^h \Delta} \tilde{k}_{t+T}^\alpha}{2} \right] \Delta$$

$$\begin{aligned} \tilde{h}_{e,t+\Delta}^4 &= (1 - \alpha)f(0)\bar{l}_e \sum_{k=1}^K \frac{\Gamma_k}{\varphi_k - g} e^{(\varphi_k - g)T} \left[ \sum_{j=0}^{n_T} e^{-\sum_{i=1}^j r_{t+(i+1)\Delta}^h \Delta} \tilde{k}_{t+(j+1)\Delta}^\alpha - \frac{\tilde{k}_{t+\Delta}^\alpha + e^{-\sum_{i=1}^{n_T} r_{t+(i+1)\Delta}^h \Delta} \tilde{k}_{t+\Delta+T}^\alpha}{2} \right] \Delta \\ &= (1 - \alpha)f(0)\bar{l}_e \sum_{k=1}^K \frac{\Gamma_k}{\varphi_k - g} e^{(\varphi_k - g)T} \left[ \sum_{j=1}^{n_T+1} e^{-\sum_{i=2}^j r_{t+i\Delta}^h \Delta} \tilde{k}_{t+j\Delta}^\alpha - \frac{\tilde{k}_{t+\Delta}^\alpha + e^{-\sum_{i=2}^{n_T+1} r_{t+i\Delta}^h \Delta} \tilde{k}_{t+\Delta+T}^\alpha}{2} \right] \Delta \end{aligned}$$

Combining the previous two expressions, we obtain

$$\tilde{h}_{e,t}^4 = (1 - \alpha)f(0)\bar{l}_e \sum_{k=1}^K \frac{\Gamma_k}{\varphi_k - g} e^{(\varphi_k - g)T} \left[ \frac{\tilde{k}_t^\alpha + e^{-r_{t+\Delta}^h \Delta} \tilde{k}_{t+\Delta}^\alpha}{2} - \frac{e^{-\sum_{i=1}^{n_T} r_{t+i\Delta}^h \Delta} \tilde{k}_{t+T}^\alpha + e^{-\sum_{i=1}^{n_T+1} r_{t+i\Delta}^h \Delta} \tilde{k}_{t+\Delta+T}^\alpha}{2} \right] \Delta + e^{-r_{t+\Delta}^h \Delta} \tilde{h}_{e,t+\Delta}^4$$

Similarly, for  $k = 1, 2, 3$ , we have

$$\begin{aligned} \tilde{h}_{e,t}^k &= (1 - \alpha)f(0)\bar{l}_e \frac{\Gamma_k}{\varphi_k - g} \left[ \sum_{j=0}^{n_T} e^{-\sum_{i=1}^j r_{t+i\Delta}^k \Delta} \tilde{k}_{t+j\Delta}^\alpha - \frac{\tilde{k}_t^\alpha + e^{-\sum_{i=1}^{n_T} r_{t+i\Delta}^k \Delta} \tilde{k}_{t+T}^\alpha}{2} \right] \Delta \\ &= (1 - \alpha)f(0)\bar{l}_e \frac{\Gamma_k}{\varphi_k - g} \left[ \frac{\tilde{k}_t^\alpha + e^{-r_{t+\Delta}^k \Delta} \tilde{k}_{t+\Delta}^\alpha}{2} - \frac{e^{-\sum_{i=1}^{n_T} r_{t+i\Delta}^k \Delta} \tilde{k}_{t+T}^\alpha + e^{-\sum_{i=1}^{n_T+1} r_{t+i\Delta}^k \Delta} \tilde{k}_{t+\Delta+T}^\alpha}{2} \right] \Delta + e^{-r_{t+\Delta}^k \Delta} \tilde{h}_{e,t+\Delta}^k \end{aligned}$$

Human wealth is then given by

$$\tilde{h}_{e,t} = h_{e,t}^4 - \sum_{k=1}^3 \tilde{h}_{e,t}^k \quad (153)$$

Aggregate human wealth:

$$\tilde{h}_t = \sum_{i=0}^{n_0} \hat{h}_{t-i\Delta,t} f(i\Delta) \Delta t + \hat{h}_{f,t} \quad (154)$$

- Law of motion of  $k_t$  and  $x_{e,t}$ :

$$\begin{aligned} \frac{\tilde{k}_{t+\Delta t} - \tilde{k}_t}{\Delta t} &= \left[ \frac{\tilde{q}_t - \bar{\Phi}_0}{\bar{\Phi}_1} - (g + \delta) \right] \tilde{k}_t \\ \frac{x_{e,t+\Delta t} - x_{e,t}}{\Delta t} &= \left[ \frac{(p_t^{id})^2}{\gamma} - \zeta_{e,t} + \frac{\hat{h}_{0,t}}{x_{e,t}} - p_t^{id} \phi \sigma_{id} \frac{\tilde{q}_t \tilde{k}_t}{\tilde{q}_t \tilde{k}_t + \tilde{h}_t} + \frac{\tilde{k}_t^\alpha - \Phi(t_t)}{\tilde{q}_t \tilde{k}_t + \tilde{h}_t} \right] x_{e,t} \end{aligned}$$

## D.1 Algorithm:

Guess a path for  $(q_t^0, p_t^{id,0})$ , then for  $l = 0, 1, \dots$ , follow

i. Solve for capital and interest rate:

$$\begin{aligned}\tilde{k}_{t+\Delta t}^l &= \left[ 1 + \left( \frac{\tilde{q}_t^l - \bar{\Phi}_0}{\bar{\Phi}_1} - (g + \delta) \right) \Delta t \right] \tilde{k}_t^l \\ r_t^l &= \frac{\alpha(\tilde{k}_t^l)^{\alpha-1} - \Phi(t_t^l)}{\tilde{q}_t^l} + \mu_A + \frac{\tilde{q}_{t+\Delta t}^l - \tilde{q}_t^l}{\Delta t} \frac{1}{\tilde{q}_t^l} + l_t^l - \delta - \gamma\sigma_A^2 - p_t^{id,l} \phi\sigma_{id}\end{aligned}$$

where  $l_t^l = \frac{\tilde{q}_t^l - \bar{\Phi}_0}{\bar{\Phi}_1}$  and  $\tilde{k}_0$  is given.

ii. Solve for consumption-wealth ratio by age and human wealth:

- Consumption-wealth ratio by age

$$\zeta_{t-a,t}^l = \left[ \sum_{k=0}^{n_a} \exp\left(-\sum_{i=0}^k \bar{r}_{t+i\Delta t}^l \Delta t\right) \Delta t + \exp\left(-\sum_{i=0}^{n_a} \bar{r}_{t+i\Delta t}^l \Delta t\right) (V^*)^{\frac{1}{\gamma}} \right]^{-1} \quad (155)$$

where

$$\bar{r}_t^l = \frac{1}{\gamma}\rho + \left(1 - \frac{1}{\gamma}\right) \left[ r_t^l + \frac{(p_t^{id,l})^2}{2\gamma} + \frac{\gamma\sigma_A^2}{2} \right] \quad (156)$$

- Human wealth

$$\begin{aligned}\tilde{h}_{f,t}^l &= (1 - \alpha)\bar{l}_f \sum_{k=0}^{n_{t,max}} \exp\left(-\sum_{i=0}^k r_{t+i\Delta t}^{h,l} \Delta t\right) \tilde{k}_{t+k\Delta t} \Delta t + \exp\left(-\sum_{i=0}^{n_{t,max}} r_{t+i\Delta t}^{h,l} \Delta t\right) \tilde{h}_{f,t,max} \\ \tilde{h}_{t-a,t}^l &= (1 - \alpha)\bar{l}_e \sum_{k=1}^K \Gamma_k e^{\varphi_k a} \sum_{j=0}^{n_a} \exp\left(-\sum_{i=0}^j r_{t+i\Delta t}^{k,l} \Delta t\right) \tilde{k}_{t+j\Delta t}^\alpha \Delta t \\ \tilde{h}_t^l &= \sum_{i=0}^{n_0} \hat{h}_{t-i\Delta t,t}^l f(a) \Delta t + \hat{h}_{f,t}^l \\ \hat{h}_{0,t}^l &= f(0)\tilde{h}_{t,t}^l / (\tilde{q}_t^l \tilde{k}_t^l + \tilde{h}_t^l)\end{aligned}$$

where  $r_t^{h,l} \equiv r_t^l + \gamma\sigma_A^2 - (g + \mu_A)$ ,  $r_t^{k,l} \equiv r_t^l + \gamma\sigma_A^2 - \mu_A - \varphi_k$ .

iii. Solve for  $(\zeta_{e,t}^l, \tilde{\omega}_{s,t}^l, x_{e,t}^l)$ :

$$\zeta_{e,t}^l = \sum_{k=0}^{n_0} \tilde{\omega}_{t-k\Delta t,t}^l \zeta_{t-k\Delta t,t}^l \Delta t \quad (157)$$

and

$$\begin{aligned}x_{e,t+\Delta t}^l &= \left[ 1 + \left( \frac{(p_t^{id,l})^2}{\gamma} - \zeta_{e,t}^l + \frac{\hat{h}_{0,t}^l}{x_{e,t}^l} - p_t^{id,l} \phi\sigma_{id} \frac{\tilde{q}_t^l \tilde{k}_t^l}{\tilde{q}_t^l \tilde{k}_t^l + \tilde{h}_t^l} + \frac{(\tilde{k}_t^l)^\alpha - \Phi(t_t^l)}{\tilde{q}_t^l \tilde{k}_t^l + \tilde{h}_t^l} \right) \Delta t \right] x_{e,t}^l \\ \tilde{\omega}_{s,t+\Delta}^l &= \begin{cases} \left[ 1 + \left( \zeta_{e,t}^l - \zeta_{s,t}^l - \frac{\hat{h}_{0,t}^l}{x_{e,t}^l} \right) \Delta t \right] \tilde{\omega}_{s,t}, & \text{for } s \in \{t + \Delta t - T, t + 2\Delta t - T, \dots, t\} \\ \tilde{\omega}_{t+\Delta-T,t+\Delta}^l + \frac{\hat{h}_{0,t+\Delta}^l}{x_{e,t+\Delta}^l}, & \text{for } s = t + \Delta t \end{cases}\end{aligned}$$

where  $x_{e,0}^l = \frac{\tilde{q}_0^l \tilde{k}_0^l + \tilde{b}_0}{\tilde{q}_0^l \tilde{k}_0^l + \tilde{h}_0^l}$ , given  $\tilde{b}_0$  and  $\tilde{\omega}_{s,0}^l$  for  $s \in \{-T, \Delta t - T, \dots, 0\}$ .

iv. Update prices using the conditions  $\tilde{q}_t^{l+1} = \tilde{q}_t^l - \eta_q \epsilon_{q,t}^l$ ;  $p_t^{id,l+1} = p_t^{id,l} - \eta_p \epsilon_{p,t}^l$ , where  $\eta_q, \eta_p > 0$  and

$$\begin{aligned}\epsilon_{q,t}^l &= \tilde{q}_t^l - \frac{1}{\tilde{k}_t^l} \left[ \frac{(\tilde{k}_t^l)^\alpha - \Phi(l_t^l) \tilde{k}_t^l}{x_{e,t}^l \tilde{\zeta}_{e,t}^l + (1 - x_{e,t}^l) \rho_f} - \tilde{h}_t^l \right] \\ \epsilon_{p,t}^l &= p_t^{id,l} - \frac{\tilde{q}_t^l \tilde{k}_t^l}{\tilde{q}_t^l \tilde{k}_t^l + \tilde{h}_t^l} \frac{\gamma \phi \sigma_{id}}{x_{e,t}^l}\end{aligned}$$

v. Stop when  $\tilde{q}_t^{l+1}$  and  $p_t^{id,l+1}$  are sufficiently close to  $\tilde{q}_t^l$  and  $p_t^{id,l}$ .

$$\frac{p_t^{id,l} x_{e,t}^l}{\gamma \phi \sigma_{id}} - \frac{\tilde{q}_t^l \tilde{k}_t^l}{\tilde{q}_t^l \tilde{k}_t^l + \tilde{h}_t^l} \tag{158}$$