

Gorman Aggregation and Welfare, Positive Representative Consumer for Prediction, Normative Representative Consumer for Welfare

14.04 Intermediate Micro Theory: Lecture 20

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Outline

- Gorman Aggregation and Welfare
- Positive Representative Consumer for prediction
- Normative representative consumer for welfare
- Indirect utility and properties of the value function
- Gorman Polar forms
- Linear expansion paths and data
- Critical review of traditional and new foundations of macroeconomics

Introduction

In this lecture, we will address the issue of aggregation of preferences. Typically, micro and (mostly) macroeconomists use models with "representative consumers" or "representative agents", in which they model aggregate phenomena as if it was a choice of a single, representative household. Why is this important?

- Gives us tractable models to understand market demand and supply (positive representative consumers)
- Allow us to do tractable welfare analysis (normative representative consumers)

Indirect Utility Function

Go again to the utility maximization problem (UMP), and define **Indirect Utility Function** $v : \mathbb{R}_+^L \times \mathbb{R}_+ \rightarrow \mathbb{R}$ as the value function of the Utility maximization problem. Namely,

$$\begin{aligned} v(p, w) &= \max_{x \in X} u(x) \\ \text{s.t.} & : px \leq w \end{aligned} \tag{1}$$

That is, $v(p, w)$ is the maximum utility attainable at the budget set $B_{p,w} \equiv \{x \in X : px \leq w\}$. It is a function of the parameters of the problem: the price vector $p \in \mathbb{R}_+^L$ and the wealth level $w > 0$. If $x(p, w)$ is the demand function, we can write

$$v(p, w) = u(x(p, w))$$

Definition (Quasi-convexity)

A function $f : X \rightarrow \mathbb{R}$ with $X \subseteq \mathbb{R}^L$ a convex set, is quasi-convex if given $x, x' \in X$ and $t \in \mathbb{R}$ such that $f(x) \leq t$ and $f(x') \leq t$, then for all $\alpha \in [0, 1]$:

$$f[\alpha x + (1 - \alpha)x'] \leq t$$

Note that a function f is quasi-convex iff and only if $-f$ is a quasi-concave function. Moreover, if f is convex, then in particular it must be also quasi-convex (so, quasi-convexity is a more general property than convexity).

Basic Properties of Indirect Utility Functions

Suppose $X \subseteq \mathbb{R}_+^L$ is a closed set, and $u(\cdot)$ is a continuous utility function representing a preference relation \succsim on X that is continuous and rational. Also, let $v(p, w)$ be the indirect utility function as defined in (1). Then

- 1 $v(p, w)$ is homogeneous of degree zero in (p, w)
- 2 $v(p, w)$ is non increasing p and non decreasing in w . If preferences are locally non satiated and $X = \mathbb{R}_+^L$ then $v(p, w)$ is strictly increasing in w
- 3 $v(p, w)$ is quasi-convex in (p, w) .
- 4 $v(p, w)$ is continuous in (p, w) for all $p \gg 0$
- 5 If $X = \mathbb{R}_+^L$, u is locally non-satiated and continuously differentiable, and $x(p, w)$ is single-valued, then $v(p, w)$ is differentiable.
- 6 **(Roy's Identity)** If $v(p, w)$ is differentiable at $(p, w) \gg 0$, then

$$x_l(p, w) = - \frac{1}{\left[\frac{\partial v(p, w)}{\partial w} \right]} \frac{\partial v(p, w)}{\partial p_l} \text{ for all } l = 1, 2, \dots, L \quad (2)$$

To prove Roy's identity, consider the Lagrangian for the consumer problem

$$\mathcal{L}[x, \lambda, (p, w)] = u(x) + \lambda_0[w - px] + \sum_{i=1}^L \lambda_i x_i$$

Let $\lambda_i^*(p, w)$ be optimal Lagrangian multiplier associated with restriction $i = 0, 1, 2, \dots, L$ when parameters are (p, w) . For simplicity, drop the dependence on (p, w) . Then, the envelope theorem implies:

$$\frac{\partial v(p, w)}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i}[x(p, w), \lambda, (p, w)] = -\lambda_0 x_i(p, w) \quad (3)$$

and

$$\frac{\partial v(p, w)}{\partial w} = \frac{\partial \mathcal{L}}{\partial w}[x(p, w), \lambda, (p, w)] = \lambda_0 \quad (4)$$

Substituting (4) into (3) gives us

$$\begin{aligned} \frac{\partial v(p, w)}{\partial p_i} &= - \left[\frac{\partial v(p, w)}{\partial w} \right] x_i(p, w) \Rightarrow \\ x_i(p, w) &= - \frac{1}{\left[\frac{\partial v(p, w)}{\partial w} \right]} \frac{\partial v(p, w)}{\partial p_i} \end{aligned} \quad (5)$$

Using in (5) the fact that v is strictly increasing in w (so $\frac{\partial v(p, w)}{\partial w} > 0$)

Preference Family: Gorman Form

Definition (Gorman Form)

Let \succsim be some rational and continuous preferences over $X \subseteq \mathbb{R}^L$. If \succsim admits a utility function $u : X \rightarrow \mathbb{R}$ for which its indirect utility function $v(p, w)$ satisfies:

$$v(p, w) = a(p) + b(p)w \quad (6)$$

we then say that the preferences are Gorman (or that they are Gorman Form).

Example: Take $X = \mathbb{R}_+^2$ and a Cobb Douglas function

$$U(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$$

so the demands are given by

$$x_1(p, w) = \alpha \frac{w}{p_1}, x_2(p, w) = (1 - \alpha) \frac{w}{p_2}$$

Then, the indirect utility function is

$$v(p, w) = \left[\alpha \frac{w}{p_1} \right]^\alpha \left[(1 - \alpha) \frac{w}{p_2} \right]^{1-\alpha} = \underbrace{0}_{a(p)=0} + \underbrace{\left(\frac{\alpha}{p_1} \right)^\alpha \left[\frac{(1 - \alpha)}{p_2} \right]^{1-\alpha}}_{b(p)} w \quad (7)$$

If α same over all household i , then coefficients $b(p)$ are the same

Aggregate demand and Wealth expansion paths

Take an economy with I agents, and let $\mathbf{w} = (w_1, w_2, \dots, w_I) \in \mathbb{R}_+^I$ be a wealth profile for the economy. Also, let $\bar{w} = \sum_{i=1}^I w_i$ the total wealth of the population, and $x^i(p, w_i)$ the demand function of agent i when wealth is w_i . We define the **aggregate demand** $\bar{x}(p, \mathbf{w})$ for this economy is the function

$$\bar{x}(p, \mathbf{w}) = \bar{x}(p, w_1, w_2, \dots, w_I) = \sum_{i=1}^I x_i(p, w_i) \quad (8)$$

The first question on aggregation is the following: **under which conditions can we ensure the existence of a aggregate demand function that depends solely on the total wealth of the population, for all possible wealth profiles w ?** More specifically, we ask the question of whether there exist a function $X : \mathbb{R}_+^L \times \mathbb{R}_+ \rightarrow \mathbb{R}_+^L$ such that:

$$\text{for all wealth profiles } w \in \mathbb{R}_+^I, \bar{x}_l(p, w) = X_l(p, \bar{w}) = X_l\left(p, \sum_{i=1}^I w_i\right) \text{ for all } l \quad (9)$$

Remarks

- This definition of aggregation does not ask that the aggregate demand function $\bar{x}(p, w)$ may come from some utility maximization problem for some utility function. It only asks that aggregate demand does not depend on the whole distribution of wealth levels, and only depends on the total wealth in the economy. **This of course is a necessary condition for the existence of a representative household.**
- On the other hand, this is a very strong restriction, because it needs to hold for all potential wealth levels $w \in \mathbb{R}_+^I$. In a GE model as the ones seen so far, not any wealth level is potentially feasible, since $w_i = p\omega_i$, where ω_i are the (fixed) endowments of the economy, and p are the prices. Therefore, this is a strong property on aggregation.

Proposition (Linear Wealth Expansion Paths)

Suppose that the individual demand functions $\{x^i(p, w_i)\}_{i=1}^l$ are differentiable. Then, if there exists a function $X(p, \bar{w})$ that satisfies (9) then for all $\mathbf{w} \in \mathbb{R}_+^l$ and all prices p :

$$\frac{\partial x_l^i(p, w_i)}{\partial w_i} = \frac{\partial x_l^j(p, w_j)}{\partial w_j} \text{ for all } i, j = 1, 2, \dots, l \text{ and } l = 1, 2, \dots, L \quad (10)$$

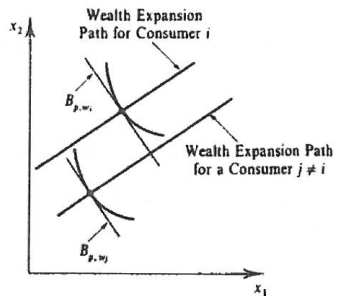


Figure 1: Parallel Linear Expansion Paths (MWG Figure 4.B.1)

Positive Representative Households and Gorman Aggregation

Definition (Positive Representative Household)

Suppose $X_i \subseteq \mathbb{R}_+^L$ and let $\{x^i(p, w_i)\}_{i=1}^I$ be the individual demand functions for a given economy, given prices p . We say that a preference \succsim^* defined on a consumption set $X^* \subseteq \mathbb{R}_+^L$ (with corresponding utility function u^*) corresponds to a Positive Representative Household iff the aggregate demand function satisfies (9) and moreover, it solves the utility maximization problem

$$\begin{aligned} X(p, \bar{w}) &= \arg \max_{x \in X^*} u^*(x) \\ \text{s.t.} & : px \leq \bar{w} \end{aligned}$$

The next Proposition states under which conditions condition (10) is satisfied, and how we could construct a positive representative household for this economy.

Proposition (Necessary and Sufficient conditions for Linear, Parallel expansion paths)

Suppose that demand functions are differentiable in w . A necessary and sufficient condition for condition (10) to be satisfied is that preferences for all agents are Gorman form, and also:

$$v_i(p, w_i) = a_i(p) + b(p) w_i \quad (16)$$

i.e. $\frac{\partial v_i(p, w)}{\partial w} = \frac{\partial v_j(p, w)}{\partial w}$ for all p, w , so that given prices, the marginal utility of wealth does not vary across households. Moreover, if we can find preferences \succsim^ that admit the indirect utility function:*

$$v^*(p, \bar{w}) = a^*(p) + b^*(p) \bar{w} \quad (17)$$

with $a^(p) = \sum_{i=1}^I a_i(p)$ and $b^*(p) = b(p)$, then \succsim^* correspond to a Positive Representative Household for this economy.*

Proof.

We will only show sufficiency (for necessity, see Deaton and Muellbauer (1980)). That is, suppose that preferences for all agents admit indirect utility functions that satisfy (16). Using Roy's identity (equation (2)) we have that

$$x_l^i(p, w_i) = -\frac{1}{\left[\frac{\partial v_i(p, w_i)}{\partial w}\right]} \frac{\partial v_i(p, w_i)}{\partial p_l} \stackrel{(i)}{=} -\left[\frac{1}{b(p)}\right] \left[\frac{\partial a_i(p)}{\partial p_l} + \frac{\partial b(p)}{\partial p_l} w_i\right] \quad (18)$$

using in (i) that $v_i(p, w)$ satisfies (16). Therefore, differentiating (18) with respect to w :

$$\frac{\partial x_l^i(p, w_i)}{\partial w_i} = -\left[\frac{1}{b(p)}\right] \left[\frac{\partial b(p)}{\partial p_l}\right] \quad (19)$$

and see that the expression in (19) does not depend either on w nor across households i , proving the desired result.

Proof (Cont).

We now want to show that if we can find preferences that admit the indirect utility function given by (17), then the preferences correspond to a Positive Representative Household for this economy. Take the indirect utility function $v^*(p, \bar{w})$, and again Roy's identity then implies that the demand for a household with these preferences will be

$$x_i^*(p, \bar{w}) = - \left(\frac{1}{\frac{\partial v^*(p, w)}{\partial \bar{w}}} \right) \frac{\partial v^*(p, w)}{\partial p_i} = - \left[\frac{1}{b(p)} \right] \left[\sum_{i=1}^I \frac{\partial a_i(p)}{\partial p_i} + \frac{\partial b(p)}{\partial p_i} \bar{w} \right] \quad (20)$$

Pulling out the summation on (20) we get

$$x_i^*(p, \bar{w}) = \sum_{i=1}^I - \left[\frac{1}{b(p)} \right] \left[\frac{\partial a_i(p)}{\partial p_i} + \frac{\partial b(p)}{\partial p_i} w_i \right] \stackrel{(i)}{=} \sum_{i=1}^I x_i^i(p, w_i) = \bar{x}_i(p, w) \quad (21)$$

using equation (18), getting the desired result.

Example Of Gorman Aggregable Preferences

Suppose household i has preferences are such that

$$X_i = \left\{ x \in \mathbb{R}_+^L : x_l \geq \underline{x}_{i,l} \right\} \text{ and}$$

$$U_i(x) = \left[\sum_{l=1}^L (x_l - \underline{x}_{i,l})^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

where the parameter $\sigma > 0$ is common across households, and the parameters $\{\underline{x}_{i,l}\}_{i=1, l=1}^{I,L}$ are the minimum subsistence levels for household i on commodity l .

Example Of Gorman Aggregable Preferences (Cont.)

It can be shown that these preferences have indirect utility function of the form:

$$v_i(p, w_i) = - \underbrace{\frac{\sum_{l=1}^L p_l x_{i,l}}{\left(\sum_{l=1}^L p_l^{1-\sigma}\right)^{\frac{1}{1-\sigma}}}}_{\equiv a_i(p)} + \left[\frac{1}{\underbrace{\left(\sum_{l=1}^L p_l^{1-\sigma}\right)^{\frac{1}{1-\sigma}}}_{\equiv b(p)}} \right] w_i \quad (22)$$

which is Gorman aggregable. The representative household's indirect utility function is:

$$v^*(p, \bar{w}) = - \frac{\sum_{l=1}^L p_l \left(\sum_{i=1}^I x_{i,l}\right)}{\left(\sum_{l=1}^L p_l^{1-\sigma}\right)^{\frac{1}{1-\sigma}}} + \left[\frac{1}{\left(\sum_{l=1}^L p_l^{1-\sigma}\right)^{\frac{1}{1-\sigma}}} \right] \bar{w}$$

Example Of Gorman Aggregable Preferences (Cont.)

Which corresponds to a representative household with preferences given by

$$U^*(x) = \left[\sum_{l=1}^L (x_l - \underline{x}_l^*)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad (23)$$

where

$$\underline{x}_l^* = \sum_{i=1}^I \underline{x}_{i,l} \quad (24)$$

i.e.; the representative household is a household with the same preferences, and a subsistence point $\underline{x}^* = (\underline{x}_1^*, \underline{x}_2^*, \dots, \underline{x}_L^*)$ which is the sum of the subsistence points of the households in the economy.

Normative Representative Household

So far, we have studied the following question:

Given a set of individual demand functions $x^i(p, w)$, can we find a fictitious household i^ such that the aggregate demand function can be thought of being generated by the choices of household i^* , which has as wealth the total wealth of the economy?*

In this section, we ask a related, but more general question in the context of General Equilibrium:

Given a set of individual demand functions $x^i(p, p\omega_i)$, can we find a fictitious household i^ such that Pareto optimal allocations in the economy can be thought as maximizing the utility of the representative household?*

Let $\mathcal{E} = \left\{ \{X_i, u_i\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega} \right\}$ be an economy with $X_i = \mathbb{R}_+^L$. We say that a household with utility $u^* : \mathbb{R}_+^L \rightarrow \mathbb{R}$ is a **Normative Representative Household** for economy \mathcal{E} if the following statements hold:

- If (x', y') Pareto dominates (x, y) , then

$$u^* \left(\sum_{i=1}^I x'_i \right) > u^* \left(\sum_{i=1}^I x_i \right)$$

- Given a feasible allocation (x, y) and an alternative feasible allocation (x', y') such that $u^* \left(\sum_{i=1}^I x'_i \right) > u^* \left(\sum_{i=1}^I x_i \right)$, then there exists an alternative feasible allocation (\tilde{x}, \tilde{y}) such that $\sum_{i=1}^I \tilde{x}_i = \sum_{i=1}^I x'_i$, $\tilde{y}_j = y'_j$ for all $j = 1, 2, \dots, J$ and (\tilde{x}, \tilde{y}) Pareto dominates (x, y) .

So, a Normative Representative Household is a fictitious household that consumes the aggregate supply of the economy, and has the property that **any Pareto optimal allocation must maximize the utility of the Normative Representative Household that consumes the aggregate net supply of the economy at that allocation.**

Definition (Norm. Rep. Household with indirect utility)

Let $\mathbf{w} = (w_1, w_2, \dots, w_l) \in \mathbb{R}_+^l$ be an income distribution, $\bar{w} \equiv \sum_{i=1}^l w_i$ and $p \in \mathbb{R}_+^L$ a price vector. Let $\{v_i(p, w_i)\}$ be indirect utility functions for agents $i = 1, 2, \dots, l$. We say that a household i^* with indirect utility function $v^*(p, \bar{w})$ is a **Normative**

Representative Household iff the following properties hold:

- 1 Let (p, \mathbf{w}) and (p', \mathbf{w}') such that $v_i(p', w'_i) \geq v_i(p, w_i)$ and $v_j(p', w'_j) > v_j(p, w_j)$ for some j . Then

$$v^* \left(p', \sum_{i=1}^l w'_i \right) > v^* \left(p, \sum_{i=1}^l w_i \right) \iff v^* (p', \bar{w}') > v^* (p, \bar{w})$$

- 2 Take (p, \mathbf{w}) and let $\bar{w} = \sum_{i=1}^l w_i$. Also, take any $p' \in \mathbb{R}_+^L$ and an aggregate income level $\bar{w}' \geq 0$ such that:

$$v^* (p', \bar{w}') > v^* (p, \bar{w})$$

Then there exists an alternative income distribution $\mathbf{w}' = (w'_1, w'_2, \dots, w'_l) \in \mathbb{R}_+^l$ such that $\sum_{i=1}^l w'_i = \bar{w}'$ and

$$v_i(p', w'_i) \geq v_i(p, w_i) \text{ and } v_j(p', w'_j) > v_j(p, w_j) \text{ for some } j$$

Normative Representative households:

Proposition

Suppose that demand functions are differentiable in w . There is a positive representative household, so if agents have Gorman form preferences, such that

$$v_i(p, w_i) = a_i(p) + b(p) w_i \quad (27)$$

Then an agent with indirect utility function $v^(p, \bar{w})$ defined as;*

$$v^*(p, \bar{w}) = \sum_{i=1}^I a_i(p) + b(p) \bar{w} \quad (28)$$

is also a Positive and a Normative representative Household for this economy.

Proof.

For the normative part, let's first show (1) in the definition. Suppose

$$v_i(p', w'_i) \geq v_i(p, w_i) \text{ and } v_j(p', w'_j) > v_j(p, w_j) \text{ for some } j \quad (29)$$

Then

$$\begin{aligned} v^* \left(p', \sum_{i=1}^I w'_i \right) &= \sum_{i=1}^I a_i(p') + b(p') \left(\sum_{i=1}^I w'_i \right) = \sum_{i=1}^I v_i(p', w'_i) \stackrel{(i)}{>} \sum_{i=1}^I v_i(p, w_i) \\ &\stackrel{(ii)}{=} v^* \left(p, \sum_{i=1}^I w_i \right) \end{aligned}$$

using in (i) condition (29) and in (ii) the definition of v^* given by (28). □

Proof (Cont.)

To prove condition (2) in the definition, suppose that for a given distribution $\mathbf{w} = (w_1, w_2, \dots, w_I)$ with $\bar{w} \equiv \sum_{i=1}^I w_i$ and for some prices p, p' and an aggregate income level \bar{w}' we have that

$$v^*(p', \bar{w}') > v^*(p, \bar{w}) \quad (30)$$

We then have to construct an alternative income distribution $\mathbf{w}' = (w'_1, w'_2, \dots, w'_I)$ such that $v_i(p', w'_i) \geq v_i(p, w_i)$ and $v_j(p', w'_j) > v_j(p, w_j)$ for some household j . To simplify the exposition, let's ignore for the moment the restriction that wealth has to be positive. □

Proof (Cont.)

Let some intuition

$$c \equiv v^*(p', \bar{w}') - v^*(p, \bar{w}) = \sum_{i=1}^I [a_i(p') - a_i(p)] + b(p') \bar{w}' - b(p) \bar{w} > 0 \quad (31)$$

Define w'_i as the level of wealth that satisfies:

$$a_i(p') + b(p') w'_i = a_i(p) + b(p) w_i + \frac{c}{I} \iff \quad (32)$$

$$w'_i = \frac{a_i(p) - a_i(p') + b(p) w_i + \frac{c}{I}}{b(p')}$$

Moreover, from (32) we can rewrite

$$v_i(p', w'_i) = v_i(p, w_i) + \frac{c}{I} > v_i(p, w_i)$$

so we showed that all agents are actually better off, showing the desired result. □

Application: Normative Representative Consumers, Pareto Optimality and Walrasian Equilibria

Take a production economy $\mathcal{E} = \left\{ \{X_i, \tilde{z}_i\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega} \right\}$ and suppose this economy admits a positive and normative representative consumer with preferences given by some utility function $u^* : X \rightarrow \mathbb{R}$. Moreover, assume (for simplicity) that

- $X_i = \mathbb{R}_+^L$
- $Y_j \subseteq \mathbb{R}^L$ are convex and admit a concave and differentiable transformation function $F_j : \mathbb{R}^L \rightarrow \mathbb{R}$; i.e. $Y_j = \{ y \in \mathbb{R}^L : F_j(y) \geq 0 \}$ (and it is therefore, closed)
- $\bar{\omega} \gg 0$
- The feasible set $\mathcal{F} = \left\{ (x, y) : x \in \mathbb{R}_+^L, F_j(y_j) \geq 0 \text{ for all } j \text{ and } x \leq \bar{\omega} + \sum_{j=1}^J y_j \right\}$ is compact.
- The utility function of the representative consumer, $u^* : X \rightarrow \mathbb{R}$ is strictly concave, differentiable and satisfies local non-satiation.
- There exist $(\tilde{x}, \tilde{y}) \in \mathcal{F}$ such that $\tilde{x} \gg 0$, $F_j(\tilde{y}_j) \geq 0$ for all $j = 1, 2, \dots, J$ and $\tilde{x} \ll \bar{\omega} + \sum_{j=1}^J \tilde{y}_j$

In the first lecture notes on the welfare theorems, we showed a partial result on this: basically, given the convexity of the utility possibility set, any Pareto optimal allocation could be represented as a maximum of a linear welfare function. Namely, we showed that if

- $X_i \subseteq \mathbb{R}_+^L$ are closed, convex sets.
- $u_i : X_i \rightarrow \mathbb{R}$ are continuous, concave functions.
- $Y_j \subseteq \mathbb{R}^L$ are convex, closed production sets

then, for any Pareto optimal allocation (x^*, y^*) there exist a vector of non-negative weights $\lambda^* \in \mathbb{R}_+^I$ such that

$$(x^*, y^*) \in \arg \max_{(x, y)} \sum_{i=1}^I \lambda_i^* u_i(x_i) \quad (25)$$

$$\text{s.t.} : \begin{cases} x_i \in X_i, y_j \in Y_j \\ \sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i + \sum_{j=1}^J y_j \end{cases}$$

Now, because there exists a normative representative household, we know that to find all (aggregate) Pareto optimal allocations, instead of solving the problem (25) and varying the vector of Pareto weights λ , we instead only need to solve the problem

$$\begin{aligned} & \max_{(x,y)} U^*(x) && (34) \\ \text{s.t. : } & \begin{cases} x_l \geq 0 \text{ for } l = 1, 2, \dots, L \\ F_j(y_j) \geq 0 \text{ for all } j = 1, 2, \dots, J \\ x \leq \bar{\omega} + \sum_{j=1}^J y_j \end{cases} \end{aligned}$$

Where x is now the aggregate consumption of the economy.

Critical Review

- See previous lecture on micro data Macro
- with micro underpinnings