Existence of Competitive Equilibria

14.04 Intermediate Micro Theory: Lecture 17

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Outline

- Fix Point Theorems
- Existence of Walrasian Prices
- Negishi Algorithm Using Second Welfare Theorem
- Gross Substitutes
- Recent Computer Science Contributions Equilibria

Mathematical Preliminaries: Fix point theorems

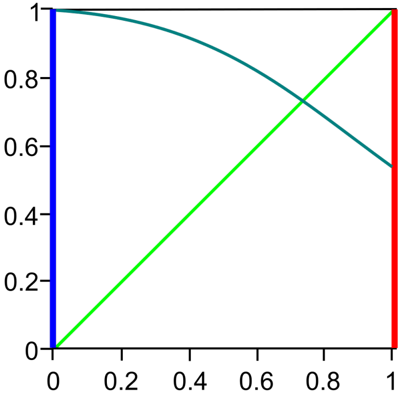
- In this section we will study the existence of solutions to general systems of nonlinear equations.
- The first thing to notice is that solving equations can be also thought of solving fixed point problems.
- A **fixed point** of a function $f : A \to A$ is an element in $x^* \in A$ such that $f(x^*) = x^*$.
- The following theorem states a very general set of conditions under which we can guarantee the existence of a fix point of a function.

Theorem (**Brower's Fix Point Theorem**)

Suppose $A \subset \mathbb{R}^L$ is non empty, convex, compact, and $f : A \to A$ is continuous. Then f has a fixed point; i.e.

$$\exists x^* \in A \text{ such that } f(x^*) = x^*.$$

Graphical proof of Brouwer fixed point theorem, in dimension 1



Kakutani's Fixed Point Theorem

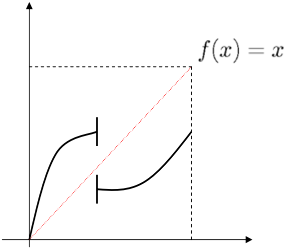
Theorem

(Kakutani) *Let A be a non-empty subset of a finite dimensional Euclidean space. Let $f : A \rightrightarrows A$ be a correspondence, with $x \in A \mapsto f(x) \subseteq A$, satisfying the following conditions:*

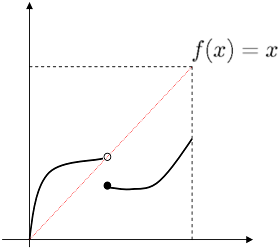
- *A is a compact and convex set.*
- *$f(x)$ is non-empty for all $x \in A$.*
- *$f(x)$ is a convex-valued correspondence: for all $x \in A$, $f(x)$ is a convex set.*
- *$f(x)$ has a closed graph: that is, if $\{x^n, y^n\} \to \{x, y\}$ with $y^n \in f(x^n)$, then $y \in f(x)$.*

Then, f has a fixed point, that is, there exists some $x \in A$, such that $x \in f(x)$.

Kakutani's Fixed Point Theorem—Graphical Illustration



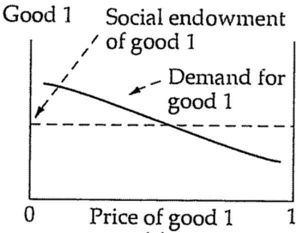
$f(x)$ is not convex-valued



$f(x)$ does not have a closed graph

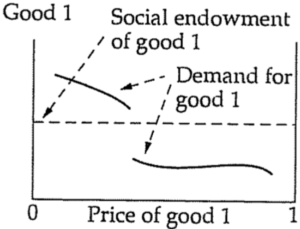
Existence of Walrasian Equilibrium: Prices

- We will find sufficient conditions to guarantee the existence of Walrasian Equilibrium.
- For ease of exposition, we focus on a pure exchange economy ($\bar{\omega} \gg 0$) with preferences that are rational, locally non satiated and continuous.
- We will represent these preferences with utility functions $u^i : X_i \to \mathbb{R}$ with $X_i = \mathbb{R}^L_+$.
- Define:
  - $x^i(p) = \arg\max \left\{ u^i(x) \text{ with } x \in \mathbb{R}^L_+ : px \leq p\omega^i \right\}$ is the demand function for consumer i with prices p and income $w = p\omega^i$.
  - $z^i : \mathbb{R}^L_+ \to \mathbb{R}^L_+ : z^i(p) = x^i(p) - \omega^i$ is the excess demand of consumer i
  - $z : \mathbb{R}^L_+ \to \mathbb{R}^L_+ : z(p) = \sum_{i=1}^{I} z^i(p)$ is the aggregate excess demand function.



Good 1
Social endowment of good 1
Demand for good 1
0
Price of good 1
1

(a)



Good 1
Social endowment of good 1
Demand for good 1
0
Price of good 1
1

(b)

Calculating Equilibrium Pareto weights: Negishi Algorithm (Cont.)

- The First Welfare Theorem implies that the set of Walrasian Equilibrium allocations is included in the set of Pareto Optimal Allocations.
- Therefore, for each Walrasian Equilibrium allocation (x^*, y^*) there must exist a vector of Pareto weights λ such that (x^*, y^*) maximizes the $\lambda-$weighted Welfare function (as we saw in the Second welfare theorem notes).
- Negishi's method relies on this fact, and instead of trying to compute the equilibrium prices for the economy, it tries to find the corresponding vector of Pareto weights.
- Once the right vector of Pareto Weights is found, one can find the Walrasian equilibrium allocation, and we can use the technique developed in the Second Welfare theorem to derive the equilibrium prices from the Pareto Optimal allocation.

Negishi's Theorem

- We will begin our approach by first noting that we can always aggregate the technology in one single representative firm, with production set $Y = \sum_{j=1}^{J} Y_j$.
- To make the notation easier, we will model an economy with only one firm (which is done, obviously, without loss of generality).
- Assume that:
  - A.1 $X_i \subseteq \mathbb{R}_+^L$ are closed, convex sets such that $0 \in X_i$ and $u_i : X_i \to \mathbb{R}$ are continuous, concave functions for all agents $i = 1, 2, \ldots, I$, which exhibit local non satiation. Define $X = \sum_{i=1}^{I} X_i$
  - A.2 $\overline{\omega} \gg 0$
  - A.3 $Y \subseteq \mathbb{R}^L$ is convex and admits a concave transformation function $Y = \left\{ y \in \mathbb{R}^L : F(y) \geq 0 \right\}$. Moreover, assume $0 \in Y$; i.e. there exist the possibility of inaction for all firms
  - A.4 The feasible set $\mathcal{F} = \{(x, y) : x_i \in X_i, y \in Y, \text{ and } \sum x_i \leq \overline{\omega} + y\}$ is compact

Negishi's Theorem

- The next theorem shows that under assumptions **(A.1)** to **(A.4)**, there exists a Walrasian Equilibrium for this economy.

Theorem (Negishi (1960))

Suppose we have a private ownership economy $\mathcal{E} = \left\{ \{X_i, u_i(\cdot), \omega_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \{\theta_{ij}\} \right\}$ *that satisfy assumptions **(A.1)** to **(A.4)**. Then, there exists a Walrasian equilibrium* (x^*, y^*, p^*) *for this economy.*

Negishi's Theorem (Cont.)

Proof

- To simplify the exposition, we will make a few extra assumptions:
  A.5 $X_i = \mathbb{R}^L_+$ and $u_i(\cdot)$ is differentiable
  A.6 $F(\cdot)$ is also differentiable
- The proof would still go through without this extra assumptions, but it would involve left and right directional derivatives, which only obscures the results.
- The whole proof can be read in Negishi (1960) or Takayama (1985).
- Since the utility functions are concave and continuous, we can replicate the construction we did in the Second Welfare theorem.
- The set of Pareto Optimal Allocations can be parametrized by the vector of Pareto weights λ.

- The Pareto problem is

$$(\hat{x}(\lambda), \hat{y}(\lambda)) \in \underset{(x,y)}{\text{argmax}} \sum_{i=1}^{I} \lambda_i u_i(x_i) \tag{7}$$

$$s.t. \; : \; \begin{cases} x_i \geq 0 & \text{for all } i, j \\ F(y) \geq 0 & \\ \sum_{i=1}^{I} x_{i,l} = \overline{\omega}_l + y_l & \text{for all } l = 1, 2, \cdots, L \end{cases}$$

Negishi's Theorem (Cont.)

Proof

- **Idea of Proof**: We know that for any $\lambda \in \Delta$ the corresponding Pareto optimal allocation $(\hat{x}(\lambda), \hat{y}(\lambda))$ is an equilibrium with transfers allocation, by virtue of the second welfare theorem.
- Therefore, we know there exist prices $\hat{p}(\lambda) = \hat{\gamma}(\lambda)$ (i.e. the Lagrange multipliers of the resource constraints) such that $[\hat{x}(\lambda), \hat{y}(\lambda), \hat{p}(\lambda), \hat{w}(\lambda)]$, with wealth levels

$$\hat{w}_i(\lambda) = \sum_{l=1}^{L} \hat{p}_l(\lambda) \hat{x}_{il}(\lambda)$$

- Now, for this allocation to be a Walrasian equilibrium (without transfers), we have the same definition as equilibrium with transfers, with the only extra constraint that

$$w_i = \sum_{l=1}^{L} p_l (\omega_{il} + \theta_i y_l)$$

Negishi's Theorem (Cont.)

Proof

- Therefore, finding an equilibrium is equivalent to finding $\lambda^* \in \Delta$ such that

$$\sum_{l=1}^{L} \hat{p}_l(\lambda^*)\,\hat{x}_{il}(\lambda^*) = \sum_{l=1}^{L} \hat{p}_l(\lambda^*)\left[\omega_{il} + \theta_i \hat{y}_l(\lambda^*)\right] \text{ for all } i = 1, \ldots I$$

  i.e. the equilibrium with transfers associated with λ^* does not need any transfers between agents!

- The proof will rely on the Kakutani's fixed point theorem, and we will need to define a mapping $f : \Delta^I \rightrightarrows \Delta^I$ to apply it to
- This mapping f will be defined in two steps

Negishi's Theorem (Cont.)

Proof

Step 1: Define a correspondence $\phi_1 : \Delta^I \rightrightarrows \Delta^I \times \mathcal{F} \times \Delta^L$ by

$$\phi_1(\lambda) = [\lambda, \hat{x}(\lambda), \hat{y}(\lambda), \hat{p}(\lambda)]$$

where $[\hat{x}(\lambda), \hat{y}(\lambda), \hat{p}(\lambda)]$ is eq. with transfers that implements $(\hat{x}(\lambda), \hat{y}(\lambda))$ as solution to Pareto problem in 7, and $\hat{p}_l(\lambda) = \hat{\gamma}_l(\lambda)$ for $l = 1 \ldots L$ are Lagrange multipliers of resource constraints (normalized to sum up to 1).

- Because of the maximum theorem, we know that ϕ_1 is an upper-hemicontinuous correspondence. *(Don't be concerned with the theorem, just accepts it's what we need).*

Negishi's Theorem (Cont.)

Proof

Step 2: since the set of feasible allocations $\mathcal{F}$ is compact, the set $\mathcal{Z} \equiv \mathcal{F} \times \Delta^I$ is also compact. Therefore there exists a number $M > 0$ such that for all $(x, y, p) \in \mathcal{Z}$ we have

$$\sum_{i=1}^{I} |\theta_i py + p\omega_i - px_i| < M$$

Given $\lambda \in \Delta^I$ and an equilibrium with transfers (x, y, p), define an updating rule $\phi_2 : \Delta^I \times \mathcal{F} \times \Delta^L \to \Delta^I$ by

$$\lambda_i' = \frac{\tilde{\lambda}_i}{\sum_i \tilde{\lambda}_i} \text{ where } \tilde{\lambda}_i = \lambda_i + \frac{\theta_i py + p\omega_i - px_i}{M} \text{ for all } i$$

If under the "guessed" eq. allocation (x, y, p) we had that $p\omega_i + \theta_i py < px_i$, then the Pareto weight is revised downward. This implies that the marginal utility of income $\mu_i^* = \frac{1}{\lambda_i}$ is greater. Concavity then implies that household i would have to expend less in consumption (providing a natural tatonnement procedure).

Negishi's Theorem (Cont.)

Proof

- Finally, define $f : \Delta^l \rightrightarrows \Delta^l$ by the composition $f = \phi_2 \circ \phi_1$
- f is non-empty, convex-valued, and upper-hemicontinuous
- Note also that the simplex $\Delta^l \subset \mathbb{R}^l$ is non-empty, compact, and convex
- Kakutani's Theorem implies that f has a fixed point $\lambda^* \in \Delta^l$
- This means that an associated price equilibrium with transfers $[\hat{x}(\lambda^*), \hat{y}(\lambda^*), \hat{p}(\lambda^*)]$ satisfies $\theta_i \hat{p}(\lambda^*)\hat{y}(\lambda^*) + \hat{p}(\lambda^*)\omega_i - \hat{p}(\lambda^*)\hat{x}_i(\lambda^*) = 0$ for all i
- This means $[\hat{x}(\lambda^*), \hat{y}(\lambda^*), \hat{p}(\lambda^*)]$ is a Walrasian equilibrium

Gross Substitutes

Definition

A Marshallian demand function $x(p, p\cdot\omega)$ satisfies gross substitutes at endowment ω if, for all prices p and p' with $p'_k > p_k$ and $p'_l = p_l$ for all $l \neq k$, we have $x_l(p', p'\cdot\omega) > x_l(p, p\cdot\omega)$ for all $l \neq k$.

- This definition of gross substitutes is a bit more subtle than the corresponding definition from consumer theory, as now increasing p_k also affects the consumers wealth. However, it is easy to see that if all goods are consumer theory gross substitutes and are also normal goods (so that demand increases with wealth), then demand functions will satisfy gross substitutes for all possible (non-negative) endowments. Gross substitutes is sufficient to generate uniqueness.

Recent CS Work

Echenique and Wierman

”The basic message that has emerged from the literature is negative: computing a Walrasian equilibrium tends to be “hard” in general settings. For example, Chen et al. (2009a) prove that finding a Walrasian equilibrium is hard (PPAD-complete and hard to approximate) even in economies with piece-wise linear concave utilities that are separable by goods. Similarly, Codenotti et al. (2006) prove that the problem is also hard in economies in which agents have Leontief preferences.
In contrast, for more limited settings, there exist positive results . . . there is a gap between the generality of the hardness results and the specificity of the instances for which finding an equilibrium is tractable. The goal of the current work is to find a middle ground where computing a Walrasian equilibrium is tractable, but which can still capture settings that include economic applications of the theory.
Our approach is different, it uses a combinatorial version of the Negishi approach for proving existence of Walrasian equilibria (Negishi 1960).”

Recent CS Work (cont.)

Paes Leme and Chiu-wai Wong (2016) “Computing Walrasian Equilibria: Fast Algorithms and Structural Properties”

”We present the first polynomial time algorithm for computing Walrasian equilibrium in an economy with indivisible goods and *general* buyer valuations having only access to an *aggregate demand oracle*, i.e., an oracle that given prices on all goods, returns the aggregated demand over the entire population of buyers. For the important special case of gross substitute valuations, our algorithm queries the aggregate demand oracle $\tilde{O}(n)$ times and takes $\tilde{O}(n^3)$ time, where n is the number of goods. At the heart of our solution is a method for exactly minimizing certain convex functions which cannot be evaluated but for which the subgradients can be computed. We also give the fastest known algorithm for computing Walrasian equilibrium for gross substitute valuations in the *value oracle model*. Our algorithm has running time $\tilde{O}((mn + n^3)T_V)$ where T_V is the cost of querying the value oracle.