Information-Constrained Optima with Retrading: An Externality and Its Market-Based Solution☆

Weerachart T. Kilenthong*[,a], Robert M. Townsend[b]

[a]*Research Institute for Policy Evaluation and Design (RIPED)*
University of the Thai Chamber of Commerce, Thailand
[b]*Massachusetts Institute of Technology, USA*

Abstract

This paper studies the efficiency of competitive equilibria in environments with a moral hazard problem and unobserved states, both with retrading in ex post spot markets. The interaction between private information problems and the possibility of retrade creates an externality, unless preferences have special, restrictive properties. The externality is internalized by allowing agents to contract ex ante on market fundamentals determining the spot price or interest rate, over and above contracting on actions and outputs. Then competitive equilibria are equivalent with the appropriate notion of constrained Pareto optimality. Examples show that it is possible to have multiple market fundamentals or price-islands, created endogenously in equilibrium.

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1. Introduction

Private information is one of the fundamental types of market imperfections. This has received much attention recently with the current financial crisis. Some in the contemporary policy debate seem to be arguing that the financial markets that suffer from private information problems cannot be efficient, even in a constrained sense; that is, improvements should be possible with enhanced regulation or government intervention. We agree that constrained efficiency may indeed fail when there are no limitations on ex post trades in spot markets. In a sense there is an externality. However, we propose a market-based solution to this particular problem.

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*Corresponding author.

Email addresses: weerachart_kil@utcc.ac.th (Weerachart T. Kilenthong), rtownsen@mit.edu (Robert M. Townsend)

We study the efficiency of competitive markets when there is a private information problem, focusing primarily on a moral hazard problem. We then extend our methods to incorporate a well known prototype, the Diamond-Dybvig model with insurance and unobserved shocks to preferences. The endogenously-created market exchanges that we model recover constrained efficiency.

Information problems have been a central concern in general equilibrium contract theory for some time. Prescott and Townsend [21, 22] show that competitive markets work efficiently, despite moral hazard problems and other ex post information problems, if one can prevent agents from retrading. As shown in [15, 1], among others, the possibility of retrade in spot markets may cause an inefficiency. As we elaborate in this paper, the interaction between private information problems and the possibility of retrade in ex post spot markets creates an "externality". More precisely, the consumption possibility set of an agent directly depends on the collective decision of all agents through the market fundamental, which determines a spot-market-clearing price. The market fundamental is, in general, determined by the distribution of resources across types of agents, with the collective decision of all agents. The impact on the consumption feasibility sets in turn influences the allocation of all agents, whenever the incentive comparability constraints of some agents are binding. More intuitively, infinitesimal agents will take the market fundamental determining prices as fixed while a (retrading-constrained) social planner takes into account the role of the collective decisions of all agents. This difference is the source of an externality.

Following Acemoglu and Simsek [1], we prove that a competitive equilibrium with moral hazard, such as in a Prescott-Townsend equilibrium, is constrained efficient when the preferences are partially separable, which implies that the marginal rate of substitution is independent of actions or efforts. In particular, the independence of marginal rates of substitution implies that a Prescott-Townsend equilibrium allocation must equate the marginal rates of substitution. Otherwise, it would be Pareto improving to do so, without violating an incentive constraint. Thus, in turn, the Prescott-Townsend equilibrium is feasible under retrading, and therefore it is constrained efficient, even with retrading. This result is a generalization of the efficiency result in [18], which proves the result for large economies with fully separable preferences. We thus identify the source of the problem, namely, with more general preferences, which are not partially separable, the Prescott-Townsend equilibrium implies the existence of agents who have different marginal rates of substitution ex post. In that case, the constrained-efficiency result fails, as discussed earlier.

We then apply a market-based solution concept, first developed in Kilenthong and Townsend [17], to internalize this externality problem. Essentially the externality can be viewed as a missing-market problem [related to the idea in Arrow 3, and his solution]. Our approach thus extends the commodity space in such a way that contracts are now contingent on market fundamentals determining spot prices as well. In other words, the lottery contracts of Prescott-Townsend are extended to include a probability distribution over future retrade prices. That is, we create markets for contracts on market fundamentals, which are the source of the problem. Allowing agents to contract ex ante on market fundamentals allows them to contract on the spot price, and internalize the externality. As a result, competitive equilibria in the extended commodity space are equivalent with retrading constrained Pareto optima under the same extended commodity space. Hence, this market-based solution cannot overcome problems from the retrading possibility completely (i.e., cannot go back to a no-retrade solution).

As shown via an example of a moral hazard problem, the retrading possibility, which causes an externality, makes it more difficult to implement the high action. Hence, there will be a larger fraction of the population taking the low action, relative to a more efficient, competitive equilibrium with price-islands. Consequently, the externality significantly lowers average output, and therefore welfare. The externality raises the marginal rate of substitution of an action-separable good for a non-separable good (the marginal rate of substitution is also the spot price between the goods, i.e., the price of an action-separable good relative to the non-separable good). A lower action means lower marginal utility of the non-separable good, and hence a higher price for the action-separable good.

We use "price-islands" to conceptualize the consistent execution of the market fundamental: A price-island specifies the composition of agents that supports the contracted price. Importantly, all price-islands, including out-of-equilibrium price-islands, are available and priced in ex ante competitive markets. A price-island is a metaphor for our assumption that agents can retrade without limitation within their own island, but not across islands. Agents cannot move, ship, or trade across islands, ex post. That is, we can separate agents into segregated exchanges. By doing so, we not only can solve the externality problem, but also can enhance ex ante welfare of the agents, relative to the optimal allocation with retrading but without price-islands. Such ex post market segregation does not get rid of the limited commitment problem completely. It also requires a registration or monitoring system to keep track of which exchange traders are supposed to be in. On the other hand, the market may endogenously choose not to segregate agents at all, and in this case there will be no need for an additional commitment (see Example 1 in Section 9).[1]

As in Prescott and Townsend [21, 22], we allow for randomized contracts. Even in the more standard setup, this eliminates potential non-convexity problems that can come with private information. We also allow agents to be assigned to price-islands at random, or to choose lotteries which implement the solution to the appropriate planning problem. It is, therefore, possible to have multiple islands with positive mass in equilibrium (see Example 2 in Section 9). On the other hand, as is shown via examples, there may well be only one price-island in equilibrium. Intuitively, it is costly to segregate agents into multiple islands because doing so reduces insurance transfers across islands to zero. This cost is larger when agents are more risk averse. Of course, it is beneficial to segregate agents into isolated islands because it limits retrading, which in turn relaxes the incentive constraint. Hence, there is a trade-off between relaxing the incentive constraints and limiting insurance transfers across agents.

Indeed, our price-islands are related to the turnpike models in Bewley [5] and Townsend [24], where agents can be viewed as spatially segregated in such a way as to limit trade. However, the islands in this paper are endogenously created in order to internalize what otherwise would be an externality, whereas the islands in the turnpike model of [24] are exogenous restrictions which allow intertemporal trade through the use of money, and there

[1] More precisely, ex-ante agents must believe that they can buy any island they want, and that trade/arbitrage across island would not be allowed. If they believe so and there is only one island in equilibrium, then it is never put to a test.

would then be autarky without that fiat money.

While this paper is closely related to Acemoglu and Simsek [1], the two papers are complementary with each other; there are, however, several differences. First, we follow [21, 22] in allowing for randomization and using a Walrasian equilibrium notion, while [1] use deterministic contracts and a Bertrand equilibrium notion [similar to 4]. Second, we propose a market-based solution concept which differs from theirs. They show that allowing firms to engage in costly monitoring over retrading markets could be welfare improving. That is in the same spirit as our no-retrading-across-islands restriction. On the other hand, their solution concept prevents agents from retrading a subset of goods whenever these goods are monitored. Our contracts only prevent agents from retrading with a subset of agents in the economy who voluntarily choose to be in different islands and do not limit in any way trading within islands. Again, both papers show that the Prescott-Townsend equilibrium is constrained efficient if preferences are partially separable, without relying on the first-order approach.

This paper is also related to a literature on pecuniary externalities that results from the possibility of retrade in spot markets, when there is some impediment to exchange [e.g., 13, 15, 16, 6, 7, 8, 2, 14, 11, 19]. As in [13, 7, 11, 19], we are explicit about the source of the externality in our context. The key difference is that our solution concept is a market-based approach that does not involve the government, while most of these papers feature government intervention.

More specifically, [e.g., 16, 2, 11] focus on retrading in an environment with a particular type of private information problem, private preference shocks, which are a standard environment in a literature on bank runs, pioneered by Diamond and Dybvig [10]. We apply our solution concept to such an environment in Section 10. We show that a competitive equilibrium with price-islands is constrained efficient under the presence of preference shocks as private information.

The remainder of this paper proceeds as follows. Section 2 describes the basic ingredients of the environment of the moral hazard model. We present the unconstrained programming problem and its corresponding Walrasian equilibrium in Section 3. In Section 4, a notion of information-constrained optimality and the Prescott-Townsend equilibrium are presented, as building blocks. We then add the key retrading friction to the Prescott-Townsend economy in Section 5. We also show that there may be an externality unless preferences are partially separable. The optimality and its decentralized equilibrium with price-islands are presented in Section 6 and Section 7, respectively. The first and second welfare theorems and an existence theorem are proved in Section 8. Section 9 discusses two numerical examples. In Section 10, an extension to the Diamond-Dybvig environment is presented, and additional heterogeneity is introduced. Section 11 concludes the paper. Appendix A contains additional proofs.

2. The Basic Environment

There are two physical commodities, labeled good 1 and good 2. For simplicity, these commodities can be produced using the sole input, called action, $a \in A \subset [\underline{a}, \bar{a}]$. For notational convenience, we use an uppercase letter to denote a set and a bold letter to denote a vector. The methods here can be easily extended to include capital.

There is a continuum of ex ante identical agents of mass 1. Each agent is endowed with the utility function $U(\mathbf{c}, a)$, where $\mathbf{c} = (c_1, c_2) \in C$ is the consumption vector of good 1 and good 2, respectively. The utility function is assumed to be differentiable, concave, increasing in $\mathbf{c}$, decreasing in a, and satisfies the usual Inada conditions with respect to $\mathbf{c}$. With an appropriate grid of consumption and increasing utility function, there will be no local satiation point in the consumption set. For simplicity, we assume that each agent is endowed with zero units of both goods.

The random production technology is given by $f(\mathbf{q}|a)$, which is the probability density function of the output vector of good 1 and good 2, $\mathbf{q} = (q_1, q_2) \in Q$, conditional on an action a taken by an agent. In other words, the probability that the realized output will be $\mathbf{q}$ is $f(\mathbf{q}|a)$ when an agent takes an action a. Thus, one can think of two subperiods: the first with the application of inputs and production; the second for output and possible retrading with final consumption. We assume for now that this production technology is the same for all agents – though this and much else can be generalized. As a probability, production satisfies

$$\sum_{\mathbf{q} \in Q} f(\mathbf{q}|a) = 1, \ \forall a \in A. \tag{1}$$

The action that an agent takes is *private information*. Hence, there is a *moral hazard* problem. The outputs are publicly observed by all parties. For simplicity, all sets, A, C, and Q, are assumed to be finite.

Given that there will be several definitions of optimality and equilibria, it is useful to summarize the important features in Table 1 below. Each row presents a notion of optimality and its corresponding equilibrium label. For notational purposes, let Z be the set of feasible "market fundamentals," which determine the spot-market-clearing prices when retrading in spot markets are possible. Its formal definition is in Section 5.

Table 1: Optimality and equilibrium notions defined in this paper.

Optimality	Decentralization	Externality	Underlying Space	Retrading
(1) Unconstrained	Walrasian equilibrium	NO	$A \times C \times Q$	N/A
(2) Information-constrained without retrading	Prescott-Townsend equilibrium	NO	$A \times C \times Q$	NO
(3) Retrading-constrained	Competitive equilibrium with retrading	YES	$A \times C \times Q$	YES
(4) Retrading-constrained with price-islands	Competitive equilibrium with price-islands	NO	$A \times C \times Q \times Z$	YES

3. The Unconstrained Economy as a Benchmark

This section presents the standard unconstrained, first-best Pareto optimal allocation and its corresponding Walrasian equilibrium. In particular, we will assume for now that there is no private information. This serves as a benchmark model for the constrained problems described later.

Without loss of generality, we will formulate the problem in the space of histograms even though there is no private information problem in this first-best world. This should also make the subsequent comparisons across regimes direct and sensible since they all are in this notation.

A contract specifies action a, and compensation in units of both goods $\mathbf{c} = (c_1, c_2)$, which is conditional on the realized output $\mathbf{q}$, i.e., $\mathbf{c}(\mathbf{q})$. Following [22], let $x(a, \mathbf{c}, \mathbf{q})$ denote a probability measure on $(a, \mathbf{c}, \mathbf{q})$. In other words, $x(a, \mathbf{c}, \mathbf{q})$ is the probability of getting a recommendation of action a, receiving compensation $\mathbf{c}$, and realizing output $\mathbf{q}$. Randomization over action a is equivalent to randomizing the contract, as any contract can be written as inducing a given action. Typically, consumption $\mathbf{c}$ is a deterministic function of output $\mathbf{q}$, which is random due to randomness in nature. With a continuum of agents, $x(a, \mathbf{c}, \mathbf{q})$ can be interpreted as the fraction of agents assigned to a contract $(a, \mathbf{c}, \mathbf{q})$. With all choice objects gridded up as an approximation, the commodity space $L \subset \mathbb{R}^n$ is assumed to be a finite n-dimensional linear space,[2] where n is the number of elements in $A \times C \times Q$.

As a probability measure, a lottery satisfies

$$\sum_{a,\mathbf{c},\mathbf{q}} x(a, \mathbf{c}, \mathbf{q}) = 1. \tag{2}$$

A feasible lottery must satisfy the following *mother-nature constraint*. This constraint ensures that the realized output $\mathbf{q}$ follows the production technology, i.e., $\sum_{\mathbf{c}} x(\mathbf{c}, \mathbf{q}|a) = f(\mathbf{q}|a)$. Using Bayes' rule, $x(\mathbf{c}, \mathbf{q}|a) = \frac{x(a,\mathbf{c},\mathbf{q})}{\sum_{\mathbf{c},\bar{\mathbf{q}}} x(a,\mathbf{c},\bar{\mathbf{q}})}$. Hence, the consistency requirement can be rewritten as:

$$f(\mathbf{q}|a) \sum_{\mathbf{c},\bar{\mathbf{q}}} x(a, \mathbf{c}, \bar{\mathbf{q}}) = \sum_{\mathbf{c}} x(a, \mathbf{c}, \mathbf{q}), \ \forall a, \mathbf{q}. \tag{3}$$

The consumption possibility set of an agent is defined by:

$$X^{fb} = \left\{x \in \mathbb{R}^n_+ : (2) \text{ and } (3) \text{ hold}\right\}. \tag{4}$$

We will use "*fb*" to denote first-best, which will distinguish it from other frictional regimes below.

The resource constraint for each good requires that the average consumption of each good be no larger than its average output:

$$\sum_{(a,\mathbf{c},\mathbf{q})} x(a, \mathbf{c}, \mathbf{q})(\mathbf{q} - \mathbf{c}) \geq \mathbf{0}. \tag{5}$$

The *unconstrained/first-best* optimal allocations are then characterized using the following Pareto planning program.

[2] The limiting arguments under weak-topology used in [21] can be applied to establish the results if L is not finite.

Program 1. (Unconstrained/First-Best)

$$\max_{x} \sum_{(a,\mathbf{c},\mathbf{q})} x(a,\mathbf{c},\mathbf{q})\, U(\mathbf{c},a) \tag{6}$$

subject to (2), (3), (5).

This is a linear program. Since X^{fb} is non-empty, compact, and convex, and the objective function is linear and continuous, a solution to the problem exists and is a global maximum. A solution to Program 1 is an unconstrained optimal allocation.

We define a corresponding Walrasian (first-best) equilibrium in the lottery space here for completeness. Needless to say, Walrasian equilibria are equivalent to unconstrained optima. Let $P(a,\mathbf{c},\mathbf{q})$ be the price of contract $(a,\mathbf{c},\mathbf{q})$.

Consumers: An agent chooses a lottery over $x(a,\mathbf{c},\mathbf{q})$ at a unit price $P(a,\mathbf{c},\mathbf{q})$ to maximize his/her expected utility

$$\max_{x\in X^{fb}} \sum_{(a,\mathbf{c},\mathbf{q})} x(a,\mathbf{c},\mathbf{q})\, U(\mathbf{c},a) \tag{7}$$

subject to the budget constraint

$$\sum_{(a,\mathbf{c},\mathbf{q})} x(a,\mathbf{c},\mathbf{q})\, P(a,\mathbf{c},\mathbf{q}) \leq 0 \tag{8}$$

taking prices $P(a,\mathbf{c},\mathbf{q})$ as given. Note that the probability and the mother-nature constraints are embedded in the agent's consumption possibility set X^{fb} as in (4). The contract is the object of interest and each contract as a bundle has a price.

Broker-Dealers: The primary role of a broker-dealer is to put together deals, i.e., buying both goods and selling insurance contracts with specified actions. In order to do so, the broker-dealer issues (sells) $y(a,\mathbf{c},\mathbf{q}) \in \mathbb{R}_+$ units of each contract $(a,\mathbf{c},\mathbf{q})$, at the unit price $P(a,\mathbf{c},\mathbf{q})$. Note that the broker-dealer can issue any non-negative number of a contract $(a,\mathbf{c},\mathbf{q})$; that is, the number of contracts issued does not have to be between zero and one and is not a lottery. It is simply the number of contracts, a real number. Let y be the vector of the number of contracts issued as one moves across feasible contracts. With constant returns to scale, the profit of a broker-dealer must be zero and the number of broker-dealers becomes irrelevant. Therefore, without loss of generality, we assume there is one representative broker-dealer, who takes prices as given.

By issuing or selling a contract $(a,\mathbf{c},\mathbf{q})$ in $\#y(a,\mathbf{c},\mathbf{q})$, in number $y(a,\mathbf{c},\mathbf{q})$, the broker-dealer will receive net transfer $\mathbf{q}-\mathbf{c}$. Given that the broker-dealer has no endowment, the production possibility requires that it needs as many goods as it delivers, or in vector notation:

$$\sum_{(a,\mathbf{c},\mathbf{q})} y(a,\mathbf{c},\mathbf{q})\,(\mathbf{q}-\mathbf{c}) \geq 0. \tag{9}$$

This constraint can also be viewed as the market clearing condition for both goods since in the "retrading period" the allocation of consumption cannot be inconsistent with $\mathbf{q}$. Formally, the production possibility set of a broker-dealer is defined by

$$Y^{fb} = \{y \in L : (9) \text{ holds}\}. \tag{10}$$

The objective of the broker-dealer is to maximize its profit by choosing y, taking prices, $P(a, \mathbf{w}, \mathbf{q})$, as given:

$$\max_{y \in Y^{fb}} \sum_{(a,\mathbf{c},\mathbf{q})} y(a, \mathbf{c}, \mathbf{q}) P(a, \mathbf{c}, \mathbf{q}) . \tag{11}$$

The existence of a maximum to the broker-dealer's problem requires that, for any bundle $(a, \mathbf{c}, \mathbf{q})$,

$$P(a, \mathbf{c}, \mathbf{q}) \leq \sum_i \widetilde{P}_i (c_i - q_i) , \tag{12}$$

where $\widetilde{P}_i \geq 0$ is the Lagrange multiplier for the feasibility constraint (9) for good i. This condition holds with equality if $y(a, \mathbf{c}, \mathbf{q}) > 0$. This condition also implies that $P(a, \mathbf{c}, \mathbf{q})$ can be negative if the contract assigns lower compensations than realized outputs, weighted by the shadow prices $\widetilde{P}_i$.

Definition 1. A *Walrasian equilibrium* is a specification of allocation (x, y), and the prices $P(a, \mathbf{c}, \mathbf{q})$ such that:

(i) for each agent, $x \in X^{fb}$ solves (7) subject to (8), taking prices $P(a, \mathbf{c}, \mathbf{q})$ as given;

(ii) for the broker-dealer, $y \in Y^{fb}$, solves (11), taking prices $P(a, \mathbf{c}, \mathbf{q})$ as given;

(iii) markets for contracts clear,

$$y(a, \mathbf{c}, \mathbf{q}) = x(a, \mathbf{c}, \mathbf{q}) , \ \forall (a, \mathbf{c}, \mathbf{q}) \in A \times C \times Q. \tag{13}$$

Note that prices $P(a, \mathbf{c}, \mathbf{q})$ come from the solution to the profit maximization problem (11). Using (13), (12) holds with equality when $x(a, \mathbf{c}, \mathbf{q}) > 0$. Then substituting (12) into (8), we end up with Program 1. Note that agents are free to retrade in ex post spot markets but whatever they can accomplish by doing so can also be done using the ex ante contracts. In particular, an optimal ex ante contract will certainly give agents the same marginal rate of substitution (which is equal to the spot price), and therefore they have no incentives to retrade.

4. An Information-Constrained Economy without Retrading: Prescott-Townsend Equilibrium

This section defines a notion of information-constrained optimality and the corresponding competitive equilibrium when there is no spot trading. This is exactly the notion defined in [22], henceforth called a Prescott-Townsend equilibrium for clarity. The essential idea is to determine constrained optimality as a solution to a programming problem. The only difference from the first-best world is that agent's action a is now private information.

The commodity space here is L, defined over $A \times C \times Q$, as in the preceding section. The probability, the mother-nature, and the resource constraints (2), (3), and (5), respectively, are as in the first-best world. With the private information on the action, a lottery must satisfy the following *incentive compatibility constraint* (IC): for each proposed a,

$$\sum_{(\mathbf{c},\mathbf{q})} x(a, \mathbf{c}, \mathbf{q}) U(\mathbf{c}, a) \geq \sum_{(\mathbf{c},\mathbf{q})} x(a, \mathbf{c}, \mathbf{q}) \frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)} U(\mathbf{c}, a') , \ \forall a'. \tag{14}$$

The left-hand side (LHS) is the expected utility from taking the recommended action a while the right-hand side (RHS) is the expected utility from taking action a'. This constraint ensures that an agent will take the recommended (possibly randomly recommended) action. The right-hand side is renormalized because the probabilities over $\mathbf{q}$ and $\mathbf{c}$ in $x(a, \mathbf{c}, \mathbf{q})$ assume action a is taken, whereas action a' is being contemplated as a deviation.

The consumption possibility set is now defined by:

$$X^{pt} = \left\{ x \in \mathbb{R}_+^n : (2), (3), \text{ and } (14) \text{ hold} \right\}. \tag{15}$$

The *information-constrained* optimal allocations without retrading are characterized using the following optimum program.

Program 2. (Information-Constrained without Retrading)

$$\max_x \sum_{(a,\mathbf{c},\mathbf{q})} x(a, \mathbf{c}, \mathbf{q}) U(\mathbf{c}, a) \tag{16}$$

subject to (2), (3), (5), (14). Alternatively, we could insert $x \in X^{pt}$ and suppress explicit reference to (2), (3), (14).

Again, this is a linear program. Since X^{pt} is non-empty, compact, and convex, and the objective function is linear and continuous, a solution to the problem exists and is a global maximum. A solution to Program 2 is an information-constrained Pareto optimal allocation without spot trading.

We now present the definition of competitive equilibria without retrading, a Prescott-Townsend equilibrium. The only difference from the Walrasian equilibrium is the presence of the IC constraint, which only affects the consumer's problem. That is, the consumption possibility set now is X^{pt} as in (15). The broker-dealer's problem is the same as in the first-best case, i.e., $Y^{pt} = Y^{fb}$. A more detailed discussion is omitted for brevity.

Definition 2. A *Prescott-Townsend equilibrium* is a specification of allocation (x, y), and the prices $P(a, \mathbf{c}, \mathbf{q})$ such that:

(i) for each agent, $x \in X^{pt}$ solves (7) with X^{pt} replacing X^{fb} subject to (8), taking prices $P(a, \mathbf{c}, \mathbf{q})$ as given;

(ii) for the broker-dealer, $y \in Y^{pt}$, solves (11), taking prices $P(a, \mathbf{c}, \mathbf{q})$ as given;

(iii) markets for contracts clear, i.e., (13) holds.

Prescott and Townsend [22] show that information-constrained Pareto optima allocations without retrading (solutions to Program 2) are equivalent to Prescott-Townsend equilibria. However, Prescott-Townsend do not allow for retrading in ex post spot markets. In principle, agents would have incentives to retrade in the spot markets if their marginal rates of substitution were different. Hence, it is useful to see if the Prescott-Townsend equilibrium allocations equalize the marginal rates of substitution. The answer is, not always. On the other hand, there is a class of preferences under which the answer is yes.

We first derive a sufficient condition under which a constrained optimal allocation equates marginal rates of substitution across agents. This condition also gives us an insight to what kind of restriction we would like to impose on preferences in order to guarantee equalization of the marginal rates of substitution. The sufficient condition is given by

$$\sum_{a'} \mu_{ic}(a,a') \frac{U_1(\mathbf{c},a')}{U_2(\mathbf{c},a)} \frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)} \left[\frac{U_2(\mathbf{c},a')}{U_1(\mathbf{c},a')} - \frac{U_2(\mathbf{c},a)}{U_1(\mathbf{c},a)} \right] = 0, \tag{17}$$

where $\mu_{ic}(a,a')$ is the Lagrange multiplier for the incentive compatibility constraint for (a,a') (14), and $U_i(\mathbf{c},e) = \frac{\partial U(\mathbf{c},e)}{\partial c_i}$ is the marginal utility with respect to good i. The formal statement and its formal proof are in Appendix A.

The sufficient condition (17) also suggests that if the marginal rates of substitution are independent of action choices, then the term in the bracket will always be zero, which implies that condition (17) will always hold. Following Acemoglu and Simsek [1], we define a class of preferences that has such a property as partially separable preferences. A utility function is said to be *partially separable* in $\mathbf{c}$ and a if

$$\frac{U_2(\mathbf{c},a)}{U_1(\mathbf{c},a)} = \frac{U_2(\mathbf{c},a')}{U_1(\mathbf{c},a')}, \ \forall a, a'. \tag{18}$$

In other words, the marginal rate of substitution does not depend on the level of effort. This class of preferences includes separable preferences. For example, $U(\mathbf{c},a) = (c_1^\rho + c_2^\rho + a^\rho)^{\frac{1}{\rho}}$, where $-\infty < \rho < 1$ and $\rho \neq 0$.

More precisely, the following proposition shows that, at the information-constrained optimal allocation without retrading, the marginal rate of substitution will be equalized. Moreover, using the welfare theorems in [22], the Prescott-Townsend equilibrium is constrained efficient and so must give all agents the same marginal rate of substitution, regardless of their actions and realized outputs. The result is summarized in Proposition 1 below. Since it is an immediate result from the sufficient condition (17), its formal proof is omitted.

Proposition 1. *If the utility function is partially separable, then an information-constrained optimal allocation without retrading equalizes marginal rates of substitution across agents, as does the corresponding Prescott-Townsend equilibrium allocation.*

5. An Information-Constrained Economy with Retrading and Externality

This section defines information-constrained optimality with retrading, and the corresponding competitive equilibrium with retrading. The only difference from the Prescott-Townsend economy is that agents now can retrade good 1 and good 2 in the spot markets after executing the contracts. This would not have harmed the Walrasian first-best allocation, but it may harm the Prescott-Townsend allocation if condition (17) fails. We will show that competitive equilibrium with retrading may not be retrading-constrained efficient; the possibility of retrading can generate an externality. Nevertheless, the (constrained) efficiency result is valid if the preferences are partially separable.

When the spot markets are available, an agent will be free to trade in the spot markets after executing her contracts, i.e., taking action a, and receiving compensation $\mathbf{c}$. In principle, we need only to keep track of the price p of good 2 for good 1 as the numeraire, but it is useful to define an economic primitive or fundamental, under which the price is the spot-market-clearing price. More formally, let z be the *market fundamental* determining the price of good 2 relative to good 1 that clears the spot markets where agents have outputs $\mathbf{q}$ and the action was a. The prices are denoted by $p(z)$. More formally, the market fundamental z is determined by the histogram of action and compensation $(a, \mathbf{c})$, which in turn depends on the collective choice of the lottery. Put differently, the market fundamental is a function of the chosen lottery x, i.e., $z(x)$.

More precisely, taking action a and compensation c as given, the agent will choose ex post in spot markets a net trade (τ_1, τ_2) to maximize utility:

$$V(\mathbf{c}, a, z) = \max_{(\tau_1, \tau_2)} U(c_1 + \tau_1, c_2 + \tau_2, a) \tag{19}$$

subject to her/his budget or spot-trade constraint

$$\tau_1 + p(z)\tau_2 = 0, \tag{20}$$

taking the spot price $p(z)$ (or the market fundamental z) as given. Notice that the indirect utility $V(\mathbf{c}, a, z)$ is a function of the market fundamental z. Let $\tau(\mathbf{c}, a, z) \equiv (\tau_1, \tau_2)$ be the vector of good 1 and good 2 that solves problem (19). Note that an individual agent has no influence on the spot price or the market fundamental.

5.1. Information-Constrained Optimality with Retrading

The commodity space here is L, defined over $A \times C \times Q$, as before. First, notice that the presence of the spot market has no effect on the probability constraint, the mother-nature constraint, or the resource constraint. That is, the probability, the mother-nature, and the resource constraints are still defined by (2), (3), and (5), respectively. This retrading possibility affects only the incentive compatibility constraint. In particular, with the presence of the spot markets, an IC constraint must take into account the possibility that agents may trade in the spot markets. As a result, it is defined in terms of the indirect utility $V(\mathbf{c}, a, z)$:

$$\sum_{(\mathbf{c},\mathbf{q})} x(a, \mathbf{c}, \mathbf{q}) V(\mathbf{c}, a, z) \geq \sum_{(\mathbf{c},\mathbf{q})} x(a, \mathbf{c}, \mathbf{q}) \frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)} V(\mathbf{c}, a', z), \ \forall a, a'. \tag{21}$$

The left-hand side (LHS) is the expected utility from taking the recommended action a and possibly trading in the spot markets. The right-hand side (RHS) is the expected utility from taking an action a' and possibly trading in the spot markets. Again, an infinitesimal agent takes the market fundamental z as given (she sees it as a fixed number) when she considers taking an an alternative action a'. On the other hand, a social planner takes into account the fact that the collective choice of lottery x affects the market fundamental; that is, the planner sees the market fundamental as $z(x)$, not just as a fixed number. This difference plays a crucial role in the existence of an externality [similar to 13, 19, 17, among others].

In addition, there is a consistency constraint ensuring that the market fundamental is z or, equivalently, that the spot market price $p(z)$ is the market-clearing price. These are actually the market-clearing constraints of the spot trades in both goods:

$$\sum_{a,\mathbf{c},\mathbf{q}} x(a, \mathbf{c}, \mathbf{q}) \tau(\mathbf{c}, a, z) = 0, \tag{22}$$

where $\tau(\mathbf{c}, a, z)$ is the spot trade function. (We will prove in Corollary 1, however, that there is no loss of generality in neglecting the consistency constraint (22), but we keep it explicit, for now.)

Definition 3. A lottery x is said to be *retrading-feasible* if it satisfies the probability constraint (2), the mother-nature constraint (3), the resource constraint (5), the IC constraint (21), and the consistency constraint (22).

We will now argue that there is no loss of generality in focusing only on lotteries with no active spot retrading, i.e., $\tau = 0$. Strictly speaking, for any retrading-feasible lottery x, there is another retrading-feasible lottery x' with no active spot trading that leads to the same consumption as under the original lottery x. This result is summarized in the following proposition. Though the proposition shows that with (21) we can consider contracts with no active trade in the spot market, one should not interpret this to mean an exogenous exclusion of these spot markets. The contracts considered here could well be the end result of holding some contracts and actively trading in the spot markets; in any event, the possibility of active retrade changes the incentive constraint and does damage.

Proposition 2. *For any retrading-feasible lottery, there is another retrading-feasible lottery with no active spot trade that generates the same consumption.*

Proof. Let x be the original lottery, which is retrading-feasible. Let $p(z)$ be the spot price given x. Suppose that lottery x is such that $x(a, \mathbf{c}, \mathbf{q}) > 0$ where $U(\mathbf{c}, a) < V(\mathbf{c}, a, z)$ for some $(a, \mathbf{c}, \mathbf{q})$; that is, the holder of the lottery will actively trade in the spot markets.

Consider an alternative contract

$$\begin{aligned} x'(a, \mathbf{c}', \mathbf{q}) &= x(a, \mathbf{c}, \mathbf{q}), \text{ when } \mathbf{c}' = \mathbf{c} + \tau(\mathbf{c}, a, z) \\ &= 0, \quad \text{otherwise,} \end{aligned} \tag{23}$$

where $\tau(\mathbf{c}, a, z)$ is the net tarde in the spot markets that solves the utility maximization problem (19) when the price is $p(z)$. A holder of this alternative contract will not trade in the spot markets by construction. It is also clear that this new compensation $\mathbf{c}'$ is equal to the net consumption under the original contract $x(a, \mathbf{c}, \mathbf{q})$. Since this is true for each and every contract, it is true for every contract as an element of L. That is, the new lottery x' and the original lottery x lead to the same consumption allocation.

We now need to check if the new lottery x' is retrading-feasible, i.e., satisfies not only (2), (3), (5), but also (21), (22). First, it is not difficult to show that it satisfies the probability constraint (2), the mother-nature constraint (3) since these constraints does not depend on the compensation. Using the consistency constraint for the original contract x, the resource constraint (5) also holds. In addition, since no one will actively trade in the spot markets under the new lottery x' at the price $p(z)$, the price $p(z)$ is the spot-market-clearing price. That is, the market fundamental is exactly z. This also implies that the consistency constraint (22) holds, by construction.

The IC constraint (21) needs special attention. Since the consumption allocation under the new lottery, $\mathbf{c}'$, also maximizes its holder's utility subject to budget constraint at the given price $p(z)$, it gives the same maximum utility as under the origin lottery x, i.e.,

$V(\mathbf{c}', a, z) = V(\mathbf{c}, a, z)$. This implies that the total value of LHS of the IC constraint (21) under the new lottery, x', is the same as under the original x.

We now need to show that it is also the case for the RHS. We utilize the fact that the indirect utility depends on the market value (at a given price) of the compensation not the compensation per se. In fact, the market value of the new compensation $\mathbf{c}'$ at the spot price $p(z)$ is given by

$$c_1' + p(z)c_2' = c_1 + \tau_1 + p(z)[c_2 + \tau_2] = c_1 + p(z)c_2 \tag{24}$$

where the last step involves (20). So, the market values of the original compensation $\mathbf{c}$ and new compensation $\mathbf{c}'$ at price $p(z)$ are the same. With the same total income, the agent will choose the same consumption, and get the same maximum utility. As a result, the RHS under x' is also the same as under x. We can now conclude that the new lottery is retrading-feasible, and leads to the same equilibrium allocation as under the original lottery x. *Q.E.D.*

Thanks to Proposition 2, we can consider only lotteries that put positive mass on contracts whose holders will optimally choose not to retrade in the spot markets, unless stated otherwise. Also, as shown in the proof of Proposition 2, the consistency constraint (22) holds automatically. Henceforth, we will drop the consistency constraint (22), unless stated otherwise. This last result is summarized in the following corollary.

Corollary 1. *The consistency constraint (22) holds for any lottery that puts positive mass on contracts that require no retrading.*

In addition, Proposition 2 implies that $V(\mathbf{c}, a, z) = U(\mathbf{c}, a)$ on the equilibrium path. As a result, the incentive compatibility constraint (21) becomes

$$\sum_{(\mathbf{c},\mathbf{q})} x(a, \mathbf{c}, \mathbf{q}) U(\mathbf{c}, a) \geq \sum_{(\mathbf{c},\mathbf{q})} x(a, \mathbf{c}, \mathbf{q}) \frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)} V(\mathbf{c}, a', z), \ \forall a, a' \in A. \tag{25}$$

We emphasize again that this IC constraint is different from the IC constraint when retrading is not permitted, (14). In particular, the market fundamental z now enters directly on the RHS of the IC constraint (25), as it affects the indirect utility of the equilibrium path. This fact also plays an important role in the existence of an externality, which will be discussed below.

The consumption possibility set of an agent with externality (ex) is defined by

$$X^{ex} = \left\{ x \in \mathbb{R}^n_+ : (2), (3), \text{and } (25) \text{ hold} \right\}. \tag{26}$$

Note that X^{ex} is nonempty, compact.

A feasible allocation now takes into explicit account the presence of the spot markets. Naturally, the IC constraint (14) is replaced by the IC constraint with spot markets (25). Hence, the Pareto program with retrading is given by the following Program .

Program 3 (Retrading-Constrained Optimality)**.**

$$\max_{x} \sum_{(a,\mathbf{c},\mathbf{q})} x(a, \mathbf{c}, \mathbf{q}) U(\mathbf{c}, a) \tag{27}$$

subject to $z = z(x)$, (2), (3),(5), and (25).

A solution to Program 3 is a *retrading-constrained* optimal allocation. We want to find that solution. Nevertheless, this program is neither linear nor convex, due to the dependence of the indirect utility on x through the market fundamental $z(x)$. Thus, the social planner implicitly chooses the market fundamental z through the choice of x of all agents. Moreover, the solution with retrading (Program 3) is typically different from, and Pareto inferior, to the solution without retrading (Program 2). This follows from the fact that the IC constraint with retrading (25) is tighter than the IC constraint without retrading (14). Nevertheless, both programs could end up being identical if the preferences are partially separable.

5.2. Competitive Equilibrium with Retrading

We now present a competitive equilibrium with retrading. The only difference from the Prescott-Townsend equilibrium is that the IC constraint is now (25), instead of (14). Again, the IC constraint affects only the consumer's problem. That is, the consumption possibility set is now X^{ex}, as in (26). The broker-dealer's problem is the same as in the first-best case, i.e., $Y^{ex} = Y^{fb}$. Detailed discussion is omitted for brevity.

Definition 4. A *competitive equilibrium with retrading* is a specification of allocation (x, y), and prices $P(a, \mathbf{c}, \mathbf{q})$ such that:

(i) for each agent, $x \in X^{ex}$ solves (7) subject to (8), taking prices $P(a, \mathbf{c}, \mathbf{q})$ as given;

(ii) for the broker-dealer, $y \in Y^{ex}$, solves (11), taking prices $P(a, \mathbf{c}, \mathbf{q})$ as given;

(iii) markets for contracts clear, i.e., (13) holds.

Note that there is no market clearing conditions for the spot markets, but this follows from the fact that we consider only contracts with no active spot trade, as discussed earlier.

5.3. The Externality

In sum there may be an externality because the consumption possibility set, as in (26), depends on the collective decision of all agents through the market fundamental $z(x)$. This dependency creates an externality. Note also that the IC constraint (25) is key to the existence of the externality because the market fundamental $z(x)$ presents in the IC constraint only.

It is also useful to illustrate the existence of an externality by comparing the optimal conditions of the programming problem and the consumer's problem in (i) of Definition 4. In particular, we will show that the optimal condition of the consumer's problem in competitive equilibrium with retrading is typically different from the necessary condition for optimality of Program 3. Though Program 3 is not a concave program, the first-order condition of Program 3 is still a necessary condition, which suffices for our purposes. For expositional reasons, we focus only on an interior solution.

For brevity, the detailed derivation is omitted. The difference between the two conditions can be written as:

$$\sum_{a,a'} \mu_{ic}(a, a') \sum_{\mathbf{c},\mathbf{q}} x(a, \mathbf{c}, \mathbf{q}) \frac{\partial V(\mathbf{c}, a', z)}{\partial z} \frac{\partial z(x)}{\partial x}, \tag{28}$$

where $\mu_{ic}(a, a')$ is the Lagrange multiplier for the incentive compatibility constraint for (a, a') (25). Naturally, we care only about allocations for which the constraints are binding. If the entire expression in (28) is zero, then a competitive equilibrium with retrading is retrading-constrained efficient. This term is typically not zero, however. Note that an infinitesimal agent takes the market fundamental, z, as invariant. To the contrary, the constrained planner can influence the market fundamental, $z(x)$, through choice of x. This key influence is the term in $\frac{\partial V(\mathbf{c},a',z)}{\partial z}\frac{\partial z(x)}{\partial x}$. Nonetheless, as shown below, this does not always have to be the case.

5.4. Partial Separability and Efficiency

This subsection shows that competitive equilibrium with retrading is retrading-constrained efficient if the utility function is partially separable. Under this assumption, the information-constrained optimality (without retrading, the second row of Table 1) coincides with the retrading-constrained optimality, see the third row of Table 1 (following from Proposition 1). The first welfare theorem in [22] then implies that the Prescott-Townsend equilibrium is also retrading-constrained efficient. Moreover, under the partial separability assumption, the Prescott-Townsend equilibrium is identical to competitive equilibrium with retrading. Therefore, competitive equilibrium with retrading is both information-constrained and retrading-constrained efficient. This result is closely related to the result in [1], and can be skipped without loss of continuity.

We only need to show that the IC constraint with spot markets (25) can be identical to the IC constraint without spot markets (14), and is identical when the preferences are partially separable. This result is summarized in Proposition 3.

Proposition 3. *If the preferences are partially separable, satisfying (18), x is a solution to Program 2 if and only if it is also a solution to Program 3.*

Proof. First, it is clear that any feasible allocation under Program 3 is feasible under Program 2, but not necessarily the other way around. As a result, a solution to Program 2 is Pareto (weakly) superior to a solution to Program 3. Therefore, we only need to show that if the solution to Program 2, x, is retrading-feasible (feasible under Program 3), then it will also be the solution to Program 3. Since the only difference between the two programs is in the IC constraint, i.e., between (14) and (25), it suffices to show that the solution to Program 2, x, also satisfies (25).

Proposition 1 proves that the marginal rates of substitution of all agents are equalized at the solution to Program 2. That is, if there were spot markets then, the spot-market-clearing price would be the same as the equalized marginal rate of substitution, denoted by $p(z)$, and there will be no active trading in the spot markets. That is, each agent's compensation maximizes her own utility subject to spot trade constraint, taking $p(z)$ and also her own action as given. This implies that the LHS of (14) is the same as the LHS of (25) given that the spot price is $p(z)$.

We now consider the RHS of the IC constraints. Partial separability implies that the solution to the utility maximization problem (19) is independent of an action choice; that is, if $\mathbf{c}$ solves (19) at a given a and $p(z)$, it must do so at the same price $p(z)$ but for any $a' \in A$. This in turn implies that $V(\mathbf{c}, a', z) = U(\mathbf{c}, a')$ if $U(\mathbf{c}, a) = V(\mathbf{c}, a, z)$, which is true for any contract $(a, \mathbf{c}, \mathbf{q})$ considered here due to Proposition 2. As a result, the RHS of (25) can be rewritten as

$$\sum_{(\mathbf{c},\mathbf{q})} x(a,\mathbf{c},\mathbf{q}) \frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)} V(\mathbf{c},a',z) = \sum_{(\mathbf{c},\mathbf{q})} x(a,\mathbf{c},\mathbf{q}) \frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)} U(\mathbf{c},a') \tag{29}$$

which is exactly the same as the RHS of (14). That is, the value of the RHS of (14) is the same as the value of the RHS of (25). Therefore, we can conclude that the solution to Program 2, x, satisfies (25), and hence retrading-feasible. *Q.E.D.*

We now summarize the implications in a lemma and a Proposition.

Lemma 1. *If the utility function is partially separable, then the Prescott-Townsend equilibrium coincides with competitive equilibrium with retrading.*

Proof. The proof is similar to the proof of Proposition 3, and therefore is omitted. *Q.E.D.*

Proposition 4. *If the preferences are partially separable, then a competitive equilibrium with retrading is retrading-constrained efficient, and a retrading-constrained optimal allocation can be decentralized as a competitive equilibrium with retrading.*

Proof. The first and second welfare theorems in [22] imply that the Prescott-Townsend equilibria are equivalent with information-constrained Pareto optima. Proposition 3 then implies that the Prescott-Townsend equilibria are equivalent with retrading-constrained Pareto optima when the preferences are partially separable. Finally, Lemma 1 implies that the competitive equilibria with retrading are equivalent with retrading-constrained Pareto optima when the preferences are partially separable. *Q.E.D.*

As discussed earlier, if the preferences are not partially separable, the above results are not valid; the competitive equilibrium with retrading may not be retrading-constrained efficient. The next section presents a market-based solution to the problem. The main idea is to extend the commodity space to include the market fundamental.

6. Internalizing the Externality: The Economy with "Price-Islands"

We internalize the externality by allowing agents to trade on the object creating the problem, the spot price $p(z)$. That is, we create rights to trading at price $p(z)$, and allow agents to buy and sell the rights. We do not rule out retrading but when agents contract to trade at $p(z)$ the collection of agents buying into that market will have to be such that with retrading the spot price will be $p(z)$. We do restrict retrade across these markets. Obviously, this requires some enforcement. So, as a metaphor, we now term a market fundamental $z \in Z$ as a *price-island*. Agents can trade freely with each other on each island ex post but cannot switch, ship, or trade goods across islands ex post. This additional commitment of not allowing trade or arbitrage across islands will not be restrictive for an economy with a single island in equilibrium (see Example 1 in Section 9) but will be restrictive if there are two or more islands in equilibrium. Again, for simplicity Z is assumed to be a finite set. We thus interpret a price-island z as a *segregated exchange* institution in which the composition of agents forms in such a way as to deliver the market fundamental z, as in [17]. We will come back to the issue of allowing only one island, which would have to be the case if we allowed arbitrage across islands with distinct prices.

Formally, the commodity space $\overline{L}$ is now extended to include the market fundamental in such a way that efficiency is restored. More formally, the commodity space is now defined over $A \times C \times Q \times Z$; that is, it is extended to include Z. Let $x(a, \mathbf{c}, \mathbf{q}, z) \geq 0$ denote a probability measure on $(a, \mathbf{c}, \mathbf{q}, z)$. In other words, $x(a, \mathbf{c}, \mathbf{q}, z)$ is the probability of receiving recommended action a, receiving consumption $\mathbf{c}$, realizing output $\mathbf{q}$, and being in island z.

6.1. The Consumption Possibility Set

The consumption possibility set is defined similarly to the case without price-islands. The probability, mother-nature, and incentive-compatibility constraints are defined by:

$$\sum_{a,\mathbf{c},\mathbf{q},z} x(a, \mathbf{c}, \mathbf{q}, z) = 1, \tag{30}$$

$$f(\mathbf{q}|a) \sum_{\mathbf{q}',\mathbf{c},z} x(a, \mathbf{c}, \mathbf{q}', z) = \sum_{(\mathbf{c},z)} x(a, \mathbf{c}, \mathbf{q}, z), \ \forall a, \mathbf{q}, \tag{31}$$

$$\sum_{\mathbf{c},\mathbf{q},z} x(a, \mathbf{c}, \mathbf{q}, z) U(\mathbf{c}, a) \geq \sum_{\mathbf{c},\mathbf{q},z} x(a, \mathbf{c}, \mathbf{q}, z) \frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)} V(\mathbf{c}, a', z), \ \forall a, a'. \tag{32}$$

Again, constraint (30) ensures that a lottery x is a probability measure. The mother-nature constraint (31) makes sure that the realized output is consistent with the production function. An agent holding $(a, \mathbf{c}, \mathbf{q}, z)$ will take a recommended action a due to the incentive constraint (32). As discussed earlier, there is no loss of generality in omitting the consistency constraint.

The consumption possibility set with price-islands (pi) is now defined by

$$X^{pi} = \left\{ x \in \overline{L} : (30), (31), \text{ and } (32) \text{ hold} \right\}. \tag{33}$$

Again X^{pi} is non-empty, compact and convex.

6.2. Retrading-Constrained Optimality with Price-Islands

The resource constraint requires that the total output be no less than the total consumption within each price-island:

$$\sum_{a,\mathbf{c},\mathbf{q}} x(a, \mathbf{c}, \mathbf{q}, z)(\mathbf{q} - \mathbf{c}) \geq \mathbf{0}, \ \forall z \in Z. \tag{34}$$

As this holds for each z, it is clear that there is no trade across price-islands. This also implies that there are no insurance transfers between islands.

A *retrading-constrained optimal allocation with price-islands* is characterized by a solution to the following programming problem.

Program 4 (Retrading-Constrained with Price-Islands)**.**

$$\max_x \sum_{a,\mathbf{c},\mathbf{q},z} x(a, \mathbf{c}, \mathbf{q}, z) U(\mathbf{c}, a) \tag{35}$$

subject to (30), (31), (32), (34).

Again, this is a linear program whose solution exists and is a global maximum given that X^{pi} is non-empty, compact, and convex, and the objective function is linear and continuous. This program allows for randomization over market fundamental z. Doing so in this economy in which all agents are ex ante identical makes it possible for there to be an equilibrium with multiple islands. That is, Program 4 is generally less constrained relative to Program 3. However, Program 4 is equivalent to Program 3 if its solution contains only one active island (see Example 1 in Section 9). Even though Program 3 and Program 4 are equivalent in this case, the competitive equilibrium with retrading (without price-islands), with the externality, does not correspond to Program 3 and is not retrading efficient.

7. Decentralization: Competitive Equilibrium with Price-Islands

The decentralized equilibrium, called competitive equilibrium with price-islands, is defined analogously to the competitive equilibrium with retrading defined in Section 5. Hence, some discussion is omitted for brevity.

Let $P(a, \mathbf{c}, \mathbf{q}, z)$ be the price of a contract $(a, \mathbf{c}, \mathbf{q}, z)$. Each agent is infinitesimally small relative to the entire economy and will take all prices as given. The broker-dealers introduced below will also act competitively.

Consumers: Each agent, taking prices, $P(a, \mathbf{c}, \mathbf{q}, z)$, as given, chooses x to maximize its expected utility:

$$\sum_{a,\mathbf{c},\mathbf{q},z} x(a, \mathbf{c}, \mathbf{q}, z)\, U(\mathbf{c}, a), \tag{36}$$

subject to the probability constraint (30), the mother-nature constraint (31), the IC constraint (32), and the ex ante budget constraint

$$\sum_{a,\mathbf{c},\mathbf{q},z} x(a, \mathbf{c}, \mathbf{q}, z)\, P(a, \mathbf{c}, \mathbf{q}, z) \leq 0. \tag{37}$$

Given that some contracts can have either positive (buying insurance) or negative (selling insurance) prices, the ex ante budget constraint (37) states that the agent both buys and sells some insurance. Prices $P(a, \mathbf{c}, \mathbf{q}, z)$ reflect premia to purchase goods, pay indemnities, receive goods, and receive trading rights. Thus, the agent will reside in island z, where she can in principle trade good 1 and good 2 at price $p(z)$ in spot markets. Also ex ante contracting can be contingent on island z. However, in the equilibrium under consideration, it will not be necessary to trade in spot markets even though they believe they could.

Broker-Dealers: Broker-dealers are similar to the ones defined in Section 5. With the price-islands, the broker-dealers need to make sure that the price-islands are consistent; that is, each price-island z must form in such a way that its market fundamental is exactly z. This type of consistency constraint is not needed, however. As discussed earlier, there is no loss of generality in considering only contracts with no active trade in spot markets. As a result, an agent in each island z will receive compensation $\mathbf{c}$ such that her marginal rate of substitution is equal to the spot price in the island $p(z)$. As a result, the market fundamental is exactly z. Therefore, the consistency constraint holds, and can be neglected.

The broker-dealer issues (sells) $y(a, \mathbf{c}, \mathbf{q}, z) \in \mathbb{R}_+$ units of each bundle $(a, \mathbf{c}, \mathbf{q}, z)$, at the unit price $P(a, \mathbf{c}, \mathbf{q}, z)$. Again, with constant returns to scale, the profit of a broker-dealer must be zero and the number of broker-dealers becomes irrelevant. Therefore, without loss of generality, we assume that there is one representative broker-dealer, which takes prices as given.

By issuing or selling a contract $(a, \mathbf{c}, \mathbf{q}, z)$, the broker-dealer will receive net transfer $\mathbf{q}-\mathbf{c}$. Given that the broker-dealer has no endowment, the production possibility requires that, for each price-island z,

$$\sum_{(a,\mathbf{c},\mathbf{q})} y\left(a, \mathbf{c}, \mathbf{q}, z\right)\left(\mathbf{q}-\mathbf{c}\right) \geq 0. \tag{38}$$

Formally, the production possibility set with price-islands is defined by

$$Y^{pi} = \left\{ y \in \overline{L} : (38) \text{ holds, for every } z \right\}. \tag{39}$$

The objective of the broker-dealer is to maximize its profit by choosing y, taking prices, $P\left(a, \mathbf{c}, \mathbf{q}, z\right)$, as given:

$$\max_{y \in Y^{pi}} \sum_{(a,\mathbf{c},\mathbf{q},z)} y\left(a, \mathbf{c}, \mathbf{q}, z\right) P\left(a, \mathbf{c}, \mathbf{q}, z\right). \tag{40}$$

The existence of a maximum to the broker-dealer's problem requires that for any contract $(a, \mathbf{c}, \mathbf{q}, z)$,

$$P\left(a, \mathbf{c}, \mathbf{q}, z\right) \leq \sum_i \widetilde{P}_i(z)\left(c_i - q_i\right), \tag{41}$$

where $\widetilde{P}_i(z) \geq 0$ is the Lagrange multiplier for the feasibility constraint (38) for good i in a price-island z. This condition holds with equality if $y\left(a, \mathbf{c}, \mathbf{q}, z\right) > 0$. Note that the shadow prices $\widetilde{P}_i(z)$ of different islands are typically different.

Market Clearing: The market-clearing conditions for lotteries are

$$x\left(a, \mathbf{c}, \mathbf{q}, z\right) = y\left(a, \mathbf{c}, \mathbf{q}, z\right), \ \forall \left(a, \mathbf{c}, \mathbf{q}, z\right). \tag{42}$$

Definition 5. A *competitive equilibrium with price-islands* is a specification of allocation (x, y), and prices $P\left(a, \mathbf{c}, \mathbf{q}, z\right)$ such that:

(i) for each agent, $x \in X^{pi}$ solves (36) subject to (37), taking prices $P\left(a, \mathbf{c}, \mathbf{q}, z\right)$ as given,

(ii) for the broker-dealer, $y \in Y^{pi}$ solves (40), taking prices $P\left(a, \mathbf{c}, \mathbf{q}, z\right)$ as given;

(iii) markets for contracts clear, i.e., (42) holds.

8. Existence and Welfare Theorems

This section proves the first and second welfare theorems and the existence of a competitive equilibrium with price-islands. In particular, we prove that a competitive equilibrium with price-islands is retrading-constrained efficient with price-islands, and a retrading-constrained optimal allocation with price-islands can be supported as a competitive equilibrium with price-islands. In addition, the existence of a retrading-constrained optimal allocation with price-islands proves the existence of a competitive equilibrium with price-islands.

We also assume that there is no local satiation point in the consumption set. This assumption is easily satisfied using reasonable specifications of the grid of consumption allocation. For example, with a strictly increasing utility function, if we include a very large consumption allocation in the grid (larger than what can be attained with output using the most productive action), then the local non-satiation assumption will be satisfied.

Assumption 1. *For any $x \in X^{pi}$, there exists $\tilde{x} \in X^{pi}$ such that*

$$\sum_{a,\mathbf{c},\mathbf{q},z} \tilde{x}\,(a,\mathbf{c},\mathbf{q},z)\,U(\mathbf{c},a) > \sum_{a,\mathbf{c},\mathbf{q},z} x\,(a,\mathbf{c},\mathbf{q},z)\,U(\mathbf{c},a). \tag{43}$$

The standard contradiction argument will be used to prove the following first welfare theorem.

Theorem 8.1. *With local non-satiation of preferences (Assumption 1), a competitive equilibrium with price-islands is retrading-constrained efficient with price-islands.*

Proof. The proof is in Appendix A. *Q.E.D.*

The Second Welfare theorem states that any retrading-constrained optimal allocation with price-islands can be supported as a competitive equilibrium with price-islands. The standard approach applies here. In particular, we will first prove that any retrading-constrained optimal allocation with price-islands can be decentralized as a compensated equilibrium with price-islands (defined below). Then, we will use a standard cheaper-point argument [see 9] to show that any compensated equilibrium with price-islands is a competitive equilibrium with price-islands. The compensated equilibrium is defined as follows. The only difference from competitive equilibrium with price-islands is the consumer's problem.

Definition 6. A *compensated equilibrium with price-islands* is a specification of allocation (x, y), and prices $P\,(a,\mathbf{c},\mathbf{q},z)$ such that

(i) for each agent, $x \in X^{pi}$ solves her expenditure minimization problem:

$$\min_{\hat{x}\in X^{pi}} \sum_{a,\mathbf{c},\mathbf{q},z} P\,(a,\mathbf{c},\mathbf{q},z)\,\hat{x}\,(a,\mathbf{c},\mathbf{q},z) \tag{44}$$

subject to

$$\sum_{a,\mathbf{c},\mathbf{q},z} \hat{x}\,(a,\mathbf{c},\mathbf{q},z)\,U\,(\mathbf{c},a) \geq \sum_{a,\mathbf{c},\mathbf{q},z} x\,(a,\mathbf{c},\mathbf{q},z)\,U\,(\mathbf{c},a) \tag{45}$$

taking prices $P\,(a,\mathbf{c},\mathbf{q},z)$ as given;

(ii) for the broker-dealer, $y \in Y$ solves (40), taking prices $P\,(a,\mathbf{c},\mathbf{q},z)$ as given;

(iii) markets for contracts clear, i.e., (42) holds.

The proof of the following theorem is a constructive proof; that is, we show that the Kuhn-Tucker conditions from Program 4 and the compensated equilibrium with price-islands are matched.

Theorem 8.2. *Any solution to the Pareto Program 4 can be supported as a compensated equilibrium with price-islands. In addition, the equilibrium expenditure is zero.*

Proof. The proof is in Appendix A. *Q.E.D.*

According to Theorem 8.2, in order to prove the second welfare theorem, we only need to show that any compensated equilibrium is a competitive equilibrium with price-islands. In particular, we will use a cheaper-point argument to show that the expenditure minimization (44) is equivalent to the utility maximization (36).

Theorem 8.3. *Any retrading-constrained optimal allocation with price-islands can be supported as a competitive equilibrium with price-islands.*

Proof. The proof is in Appendix A. *Q.E.D.*

We will now show that the competitive equilibrium with price-islands exists. As discussed earlier, given that the feasible set of Program 4 is non-empty, compact, and convex, and its objective function is continuous, a solution to the Program exists. Using the second welfare theorem (Theorem 8.3), the solution is a competitive equilibrium with price-islands. Therefore, we can conclude that a competitive equilibrium exists. This result is summarized in Theorem 8.4. Note that Negishi's mapping method [Negishi 20] would be needed if there were ex ante heterogenous agents [see 23, as an example].

Theorem 8.4. *A competitive equilibrium with price-islands exists.*

8.1. Possibility of Multiple Active Price-Islands

When we rule out trade across the price-island, we are, to be blunt, going against the original problem, and now we prevent some forms of retrading. Actually our assumption is a bit different from simply restricting trade a priori, since agents can choose (subject to budget constraints, of course) to retrade in any particular island. We still allow that retrade, the original source of the problem. Further, all agents may choose to trade in the same island, as in Environment 1 in Section 9. Then there is no temptation to arbitrage ex post. Still, even when there is only one island in equilibrium, agents believe that when they choose islands ex ante they have various distinct prices that they cannot retrade across islands ex post. Presumably, this belief is easier to instill ex ante, via costly threats that never need to be implemented if the threats are credible. (Operationally this would seem to be easier than preventing ex post arbitrage). Thus, it is important to see if we can constrain agents to be in one island a priori.

We show that if we require agents to (possibly randomly) choose a desired price-island z before they choose everything else, then there will always be a unique active price-island in equilibrium. To be more precise, we require that the probability of z be independent of $(a, \mathbf{c}, \mathbf{q})$, i.e., for any price-island z, $Pr\left(z|a, \mathbf{c}, \mathbf{q}\right) = Pr\left(z|a', \mathbf{c}', \mathbf{q}'\right)$ for any $(a, \mathbf{c}, \mathbf{q})$, $(a', \mathbf{c}', \mathbf{q}') \in A \times C \times Q$. Using Bayes' rule, this condition can be written in term of x as, for any z :

$$\frac{x\left(a, \mathbf{c}, \mathbf{q}, z\right)}{\sum_z x\left(a, \mathbf{c}, \mathbf{q}, z\right)} = \frac{x\left(a', \mathbf{c}', \mathbf{q}', z\right)}{\sum_z x\left(a', \mathbf{c}', \mathbf{q}', z\right)}, \forall \left(a, \mathbf{c}, \mathbf{q}\right), \left(a', \mathbf{c}', \mathbf{q}'\right) \in A \times C \times Q. \tag{46}$$

With these constraints, Program 4 becomes the following.

Program 5.

$$\max_x \sum_{a,\mathbf{c},\mathbf{q},z} x\left(a,\mathbf{c},\mathbf{q},z\right)U(\mathbf{c},a) \tag{47}$$

subject to (30), (31), (32), (34), (46).

It is clear that the constrained set is non-empty and compact, and the objective function is convex and continuous. However, the constrained set is non-convex due to the presence of constraint (46). Nevertheless, a solution to the problem exists. Put differently, there exists a retrading-constrained optimal allocation with a single island, which is defined as a solution to Program 5.

We will now prove that constraint (46) guarantees the uniqueness of the islands. In particular, we will show that any feasible lottery satisfying (46) must put all mass in one and only one island. The result is summarized in Proposition 5. Roughly speaking, because agents choose price-islands before everything else, the distribution of $(a, \mathbf{c}, \mathbf{q})$ within each active price-island must be the same, due to the Law of Large Numbers (since there is a continuum of agents). As a result, they will end up with the same market fundamental. Hence, there can be only one active price-island.

Proposition 5. *Any feasible lottery x, which satisfies (30), (31), (32), (34), and (46) will put positive mass on one z only, i.e., if $\sum_{a,\mathbf{c},\mathbf{q}} x\left(a,\mathbf{c},\mathbf{q},z\right) > 0$, then for any $z' \neq z$, $\sum_{a,\mathbf{c},\mathbf{q}} x\left(a,\mathbf{c},\mathbf{q},z'\right) = 0$. In addition, a solution to Program 5 puts positive mass on one z only.*

We now turn to the corresponding competitive equilibrium, which can be defined analogously to competitive equilibrium with price-islands. The formal definitions and proofs are omitted for brevity. The only difference is that the consumption possibility set is now subject to an additional constraint (46), which is neither linear nor convex. Nevertheless, we can still impose a local non-satiation assumption similar to Assumption 1. Given the local non-satiation, the competitive equilibrium with single island is retrading-constrained optimal; that is, the first welfare theorem still holds.

The first welfare theorem could be vacuous, however, since the competitive equilibrium may not exist due to the non-convexity of the consumption possibility set, created by (46). As in standard general equilibrium models, the non-convexity may overturn the continuity of the demand function, which is required for the application of the Kakutani fixed-point theorem. Loosely speaking, the non-convexity may induce a "jump" in that demand function. In principle, the discontinuity problem can be "aggregated out" and the existence theorem can be proved in an economy with a continuum of ex ante heterogeneous types, as in [25]. Unfortunately, adding ex ante heterogeneity causes another problem, as is discussed below.

The non-convexity of the consumption possibility set also causes a difficulty with proving the second welfare theorem. In particular, the proof employed elsewhere in this paper is not applicable because the Kuhn-Tucker conditions are necessary but may not be sufficient without the convexity.

9. Numerical Examples

This section presents numerical examples of various environments and for each Walrasian equilibrium, Prescott-Townsend equilibrium, and competitive equilibrium with price-islands.

The first environment is illustrative of an economy in which there is a unique active price-island in competitive equilibrium with price-islands. That is, all agents end up in only one price-island in an equilibrium, even though many price-islands are feasible for trade. On the other hand, the second environment presents an economy in which there are multiple active price-islands in the competitive equilibrium with price-islands. The only difference in term of primitives between the two environments is the relative risk aversion. In particular, the representative agent in the first environment is more risk-averse. This suggests intuitively that it is more likely for there to be one active price-island if agents care most about insurance.

To understand this conjecture, we first ask if we can implement an information-constrained optimal allocation (without retrading) by segregating agents with different marginal rates of substitution into different islands. The answer is no, in general.

An information-constrained optimal allocation (Prescott-Townsend) typically requires transfers of resources between agents with different realized outputs. These insurance transfers are not possible when these agents are in different isolated islands, by construction. This is the cost of segregating agents into multiple islands. On the other hand, it is beneficial to segregate agents into isolated islands because this limits retrading, which in turn relaxes the IC constraint. In conclusion, there is a trade-off between relaxing the IC constraints and limiting insurance transfers across islands. As the following examples will show, the gain from insurance dominates, and ruling out some insurance is more costly when agents are more risk averse.

The first example also presents a competitive equilibrium with retrading. The retrading possibility makes it more difficult to implement the high action. Hence, there will be a larger fraction of the population taking the low action, relative to a more efficient, competitive equilibrium with price-islands. In addition, the externality significantly lowers average output, and therefore welfare. The externality raises the marginal rate of substitution of the action-separable good for the non-separable good (the marginal rate of substitution is also the spot price between the goods, i.e. the price of the action-separable good relative to the non-separable good). A lower action means lower marginal utility of the non-separable good, hence a higher price for the action-separable good.

Environment 1. There are two possible actions, $a \in \{1, 3\}$. The production technology is summarized in Table 2.

Table 2: Production Technology.

Outputs: $\mathbf{q}$		Probability: $f(\mathbf{q}\|a)$	
q_1	q_2	$f(\mathbf{q}\|a=1)$	$f(\mathbf{q}\|a=3)$
0.10	0.10	0.95	0.05
1.00	1.00	0.05	0.95

Note that there are two possible outputs. In addition, the outputs of good 1 and good 2 are perfectly correlated, moving in tandem; that is, the output of good 1 is high when the output of good 2 is high, and vice versa. Each agent's utility function is given by:

$$U(\mathbf{c}, a) = a^{\sigma} \frac{c_1^{1-\gamma}}{1-\gamma} + \frac{c_2^{1-\gamma}}{1-\gamma}, \tag{48}$$

where in this environment $\sigma = 2$ and $\gamma = 2.5$. Thus agents are risk averse. However, action and good 1 are complements in utility, whereas this is not true for good 2. In particular, the marginal utility with respect to good 1 is positive and strictly increasing in action a. Note also that the marginal utility with respect to action is negative, given that $1 - \gamma < 0$. That is, holding consumption fixed, a higher action implies a lower utility.

The first-best optimal allocation, which is also a Walrasian equilibrium, assigns the low action $a = 1$ with probability 0.077, and the high action $a = 3$ with probability 0.923. The randomization over action, even in the first best, is due to the discreteness of the action choice.[3] In addition, an agent taking the low action receives consumption $\mathbf{c} = (0.3881, 0.8926)$, and an agent taking the high action receives consumption $\mathbf{c} = (0.9347, 0.8926)$, regardless of realized outputs $\mathbf{q}$. Note that under separable preferences consumption would be the same regardless of a and $\mathbf{q}$, but the nonseparability creates an interaction. This first-best allocation gives an agent expected utility of -7.1310. In addition, the marginal rates of substitution of all agents are identical and equal to 0.1247, which is the spot-market-clearing price in this case.

The Prescott-Townsend equilibrium allocation[4] (a solution to Program 2) is summarized in Table 3. The expected utility of an agent at the Prescott-Townsend equilibrium is -7.3856, which is lower than the first-best outcome, as anticipated.

Table 3: Prescott-Townsend equilibrium allocation: traded contracts. Each column is a traded contract/bundle of $(a, \mathbf{c}, \mathbf{q})$ in Prescott-Townsend equilibrium.

	traded contracts by ex post types			
	type 1: low action		type 2: high action/low output	type 3: high action/high output
a	1.0000	1.0000	3.0000	3.0000
c_1	0.3767	0.3767	0.8850	0.9257
c_2	0.8732	0.8732	0.2003	0.8910
q_1	0.1000	1.0000	0.1000	1.0000
q_2	0.1000	1.0000	0.1000	1.0000
MRS	0.2833	0.2833	1.0319	0.1177
$x(a, \mathbf{c}, \mathbf{q})$	0.1133	0.0060	0.0440	0.8368

We now consider in more detail the equilibrium allocation presented in Table 3 as ex post endowment profiles of agents. More precisely, action a and compensation $\mathbf{c}$ define agent ex post types since this is the relevant information regarding the (ex post) spot markets. Table 3 shows that there are three (ex post) types of agents: (i) type 1 (low type) takes action $a = 1$ and receives compensation $\mathbf{c} = (0.3767, 0.8732)$; (ii) type 2 (high action but low output type) takes action $a = 3$ and receives compensation $\mathbf{c} = (0.8850, 0.2003)$; (iii) type 3 (high action

[3]It particular, the most important thing is the wedge between the cost of the low and high actions. For example, if the available actions were $\{1, 2\}$, then the first-best allocation would have assigned only the high action. Nevertheless, as shown in [12], it could be optimal depending on preferences and technology to randomize over actions even if the action choice set is continuous.

[4]In this example, we solve the linear programming problem using CPLEX on a server at age3.uchicago.edu. In this example we grid up the consumption allocation, $c_i \in [0.01, 1]$, into 1000 points.

and high output type) takes action $a = 3$ and receives compensation $\mathbf{c} = (0.9257, 0.8910)$. Type 1 agents receive constant compensation regardless of their outputs; that is, they are fully insured against the output shocks, as in the Walrasian equilibrium. On the other hand, conditional upon taking high action, high types 2 and 3 are not fully insured. In particular, they will receive a lower compensation of good 2 if their outputs are unluckily low but will receive a high compensation of both goods if their outputs are luckily high. Interestingly, the main difference in the compensation between lucky and unlucky high type agents is in the second good. This is due to the fact that the marginal utility with respect to good 1 depends on the action level while the marginal utility with respect to good 2 does not. Therefore, given the high action, it will be too costly to make the compensation in terms of good 1 too low.

In addition, the marginal rates of substitution of an agent type 1, an agent type 2, and an agent type 3 are 0.2833, 1.0319, 0.1177, respectively, which are clearly different. Therefore, the Prescott-Townsend equilibrium is not valid with retrading. In particular, suppose that the spot markets were available. With different marginal rates of substitution, agents would trade in spot markets. Each agent will choose consumption (by trading in the spot markets) to maximize her own utility subject to budget constraints, taking the spot price as given. The ex post spot price in this case with the "surprise" of allowing spot retrade would have been $p(z) = 0.1330$.

The competitive equilibrium with retrading, with its externality, is summarized in Table 4. The (ex post) spot market price between good 2 (action-separable one) and good 1 (non-separable one) in this case is $p(z) = 0.3545$, which is higher than the spot market price with "surprise" retrade (0.1330), and higher than the spot price in competitive equilibrium with price-islands (0.1373), which is presented below. That is, the retrading possibility with its externality causes the spot price to be too high. In addition, it causes welfare losses. The ex ante expected utility of an agent at the competitive equilibrium with retrading is -10.2129, which is significantly lower than the expected utility in the Prescott-Townsend equilibrium (-7.3856). As shown below, introducing price-islands reduces the spot price, and also raises the expected utility. The possibility of retrade in the spot markets also affects the action choices. In particular, there is a larger fraction of population, $\sum_{\mathbf{c},\mathbf{q},z} x\left(a = 1, \mathbf{c}, \mathbf{q}, z\right) = 0.6947$, taking the low action in the competitive equilibrium with retrading relative to the Prescott-Townsend equilibrium (0.1192), and larger than the competitive equilibrium with price-islands (0.1401) (moreover, only 0.0770 fraction of the population take the low action in the first-best world). This labor input difference causes average outputs to be different. The average outputs in competitive equilibrium with retrading are now 0.3923 for both goods. These are lower than the average outputs in the Prescott-Townsend equilibrium and in the competitive equilibrium with price-islands, which are 0.8584 and 0.8397 for both goods, respectively.

The competitive equilibrium with price-islands[5] (a solution to Program 4) is summarized in Table 5. Though many price-islands are available for trade, there is *only one active price-*

[5] We solve the linear programming problem using CPLEX on a server at age3.uchicago.edu. In this example, we grid up the market fundamentals into $p(Z) = \{0.0873, 0.1373, 0.1873, 0.2, 0.25, 0.3, 49.5, 50, 50.5\}$. Those few large numbers are included to ensure that the solution will not be stuck at the corner of the grids. We also grid up the consumption allocation, $c_i \in [0.1, 1]$, into 10,000 points.

Table 4: Competitive equilibrium allocation with retrading, with its externality: traded contracts. Each column is a traded contract/bundle of $(a, \mathbf{c}, \mathbf{q})$ in competitive equilibrium with retrading. The (ex-post) spot price is $p(z) = 0.3545$.

	traded contracts by ex post types			
	type 1: low action		type 2: high action/low output	type 3: high action/high output
a	1.0000	1.0000	3.0000	3.0000
c_1	0.2368	0.2368	0.1137	0.7794
c_2	0.3585	0.3585	0.0715	0.4900
q_1	0.1000	1.0000	0.1000	1.0000
q_2	0.1000	1.0000	0.1000	1.0000
$x(a, \mathbf{c}, \mathbf{q})$	0.6600	0.0347	0.0152	0.2900

island with spot price $p(z) = 0.1373$. The expected utility of an agent at this competitive equilibrium with price-islands is -7.5302, which is less than the expected utility in the Prescott-Townsend equilibrium (-7.3856).

Table 5: Competitive equilibrium with price-islands allocation: traded contracts. Each column is a traded contract/bundle in the competitive equilibrium with price-islands.

	traded contracts by ex post types in island $p(z) = 0.1373$			
	type 1: low action		type 2: high action/low output	type 3: high action/high output
a	1.0000	1.0000	3.0000	3.0000
c_1	0.3745	0.3745	0.8410	0.9222
c_2	0.8289	0.8289	0.7729	0.8475
q_1	0.1000	1.0000	0.1000	1.0000
q_2	0.1000	1.0000	0.1000	1.0000
$p(z) = MRS$	0.1373	0.1373	0.1373	0.1373
$x(a, \mathbf{c}, \mathbf{q}, z)$	0.1338	0.0063	0.0411	0.8160

The next environment illustrates a competitive equilibrium with *multiple active price-islands*. The only difference from the previous environment is that the agents are now less risk averse. This is again consistent with the conjecture, discussed earlier, that multiple price-islands are more likely when agents are less risk-averse.

Environment 2. The primitives in this environment are the same as in the previous one except for the utility function. More precisely, each agent's utility function is given by (48), with $\sigma = 2$ and $\gamma = 2$. The only difference is in $\gamma = 2$, as it was 2.5 in the previous example. That is, the agents in this example are less risk-averse.

Again, given that agents are risk-averse and ex ante identical, the first-best optimal allocation, which is also a Walrasian equilibrium, assigns the low action $a = 1$ with probability 0.5848, and high action $a = 3$ with probability 0.4152. In addition, an agent taking the low action will receive consumption $\mathbf{c} = (0.2630, 0.4813)$, and an agent taking the high action will receive consumption $\mathbf{c} = (0.7889, 0.4813)$, regardless of realized outputs $\mathbf{q}$. This allocation gives an agent an expected utility of -9.0386. In addition, the marginal rates of substitution of all agents are identical and equal to 0.2985, which is the spot-market-clearing price in this case.

The Prescott-Townsend equilibrium allocation[6] (a solution to Program 2) is summarized in Table 6. The expected utility of an agent at the Prescott-Townsend equilibrium is -9.2296, which is lower than the first-best outcome, as anticipated.

Table 6: Prescott-Townsend equilibrium allocation: traded contracts. Each column is a traded contract/bundle of $(a, \mathbf{c}, \mathbf{q})$ in Prescott-Townsend equilibrium.

	traded contracts by ex post types			
	type 1: low action		type 2: high action/low output	type 3: high action/high output
a	1.0000	1.0000	3.0000	3.0000
c_1	0.2558	0.2558	0.7423	0.7859
c_2	0.4668	0.4668	0.0744	0.4787
q_1	0.1000	1.0000	0.1000	1.0000
q_2	0.1000	1.0000	0.1000	1.0000
MRS	0.3001	0.3001	11.0574	0.2995
$P(a, \mathbf{c}, \mathbf{q})$	4.0746	-13.8626	9.7322	-5.6781
$x(a, \mathbf{c}, \mathbf{q})$	0.5765	0.0303	0.0197	0.3735

In this case, there are three (ex post) types of agents, similar to the first environment. In addition, the marginal rates of substitution of agent type 1, agent type 2, and agent type 3 are 0.3001, 11.0574, 0.2995, respectively, which are clearly different. Therefore, the Prescott-Townsend equilibrium is not valid with retrading. In addition, the ex post utility of type 1, type 2, and type 3 are $-6.0519, -25.5620, -13.5400$, respectively. We will compare these ex post utility values to the ones in the competitive equilibrium with price-islands later.

The competitive equilibrium with price-islands[7] (a solution to Program 4) is summarized in Table 7.

There are *two active price-islands* with spot price $p(z) = 0.3175$ and $p(z) = 0.4250$. The first island, $p(z) = 0.3175$, consists of all agents who take high action and receive high output, type 3' s, and some of the agents who take low action regardless of the output, type 1' s. The second island, $p(z) = 0.4250$, consists of all agents who take high action but receive low output, type 2' s, and some of the agents who take low action regardless of the output, type 1' s.

We can gain more intuition about how the islands are formed as such by comparing this result to the Prescott-Townsend equilibrium. Recall that the Prescott-Townsend equilibrium is Pareto superior to the competitive equilibrium with price-islands. Hence, it is optimal to

[6] In this example, we solve the linear programming problem using CPLEX on a server at age3.uchicago.edu. In this example we grid up the consumption allocation, $c_i \in [0.01, 1]$, into 1000 points.

[7] The programming problem is solved using CPLEX on a server at age3.uchicago.edu. In this example, the market fundamentals are $p(Z) = \{0.2675, 0.3175, 0.3675, 0.375, 0.425, 0.475, 49.5, 50, 50.5\}$. Those few large numbers are included to ensure that the solution will not stuck at the corner of the grids. We also grid up the consumption allocation, $c_i \in [0.1, 1]$, into 10,000 points.

Table 7: Competitive equilibrium with price-islands allocation: traded contracts. Each column is a traded contract/bundle in the competitive equilibrium with price-islands. There are two active price islands, $p(z) = 0.3175$ and $p(z) = 0.4250$.

	traded contracts by ex post types					
	in island $p(z) = 0.3175$			in island $p(z) = 0.4250$		
	type 1		type 3	type 1		type 2
a	1.0000	1.0000	3.0000	1.0000	1.0000	3.0000
c_1	0.2527	0.2527	0.7842	0.2671	0.2671	0.6791
c_2	0.4486	0.4486	0.4639	0.4096	0.4098	0.3472
q_1	0.1000	1.0000	1.0000	0.1000	1.0000	0.1000
q_2	0.1000	1.0000	1.0000	0.1000	1.0000	0.1000
$p(z)$	0.3175	0.3175	0.3175	0.4250	0.4250	0.4250
$P(a, \mathbf{c}, \mathbf{q}, z)$	4.1136	-14.5204	-6.0245	4.1954	-14.6733	10.4656
$x(a, \mathbf{c}, \mathbf{q}, z)$	0.5726	0.0135	0.3583	0.0193	0.0166	0.0154

keep the allocation with price-islands as "close as possible" to the former one (formally of course utility is the metric, but the former allocations are some sense the target). Note that type 1 (0.3001) and type 3 (0.2995) have low and similar marginal rates of substitution in the Prescott-Townsend equilibrium. Therefore, it is optimal to keep them together in a low price-island. On the other hand, type 2 agents have a very high marginal rate of substitution in the Prescott-Townsend equilibrium, and these agents are insurance receivers, who receive more of each good than is produced by themselves. Hence, there must be insurance providers, which in this case are some of the type 1 agents.

The expected utility of an agent at this competitive equilibrium with price-islands is -9.3632, which is less than the expected utility in the Prescott-Townsend equilibrium (-9.2296). The ex post utility of type 1, type 2, and type 3 are -6.1860, -16.1323, -13.6330, respectively. These ex post utility values are much closer to each other relative to the values in the Prescott-Townsend equilibrium. Put differently, ex post inequality in the competitive equilibrium with price-islands is lower relative to the Prescott-Townsend equilibrium.

10. Extensions

This section discusses two extensions: (i) unobserved states as private information; and (ii) ex ante heterogeneity.

10.1. Unobserved States as Private Information

This section illustrates how to apply our solution method to solve an externality in an economy with unobserved states or preference/liquidity shocks and spot markets [e.g., 16, 2, 11]. Similar to the moral hazard problem, if there were no spot markets, then the Prescott-Townsend equilibria would have been equivalent to Pareto optima. However, this liquidity problem features an externality if agents can trade in spot/private markets ex post due to the interaction of binding incentive constraints and the spot prices. For brevity, the discussion of the existence of an externality will be omitted. We will focus on how to apply our solution method with price-islands here. As in [22] and [11], we focus only on incentive compatible allocations (no sequential service constraint). Hence, there are no bank runs in this model.

There are three periods, $t = 0, 1, 2$, and one physical commodity in each period. There is a continuum of ex ante identical agents with total mass one. Each of them is endowed with e units of the good in the contracting period, $t = 0$. Following the literature, there are two technologies or assets. First, the short-term asset is a storage technology, whose return from t to $t + 1$ is 1, i.e., saving one unit of the good today will return one unit of the good tomorrow. The second asset is the long-term asset. The long-term investment must be taken at $t = 0$, and its return $R > 1$ will be realized at $t = 2$.

Let $\theta \in \Theta$ be a preference/liquidity shock. The shock is drawn at $t = 1$ with $\pi(\theta)$ as the probability that an agent will receive θ shock, and $\sum_\theta \pi(\theta) = 1$. With the continuum of agents, we also interpret $\pi(\theta)$ as the fraction of agents receiving θ shock. The utility function conditional on a shock θ is given by $U(\mathbf{c}, \theta)$, where $\mathbf{c} = (c_1, c_2)$ is the vector of consumption allocations in both periods. For example, in the Diamond-Dybvig model, the shock will dictate if an agent would like to consume now or later. As before, the utility function is assumed to be differentiable, concave, increasing in $\mathbf{c}$, and satisfies the usual Inada conditions with respect to $\mathbf{c}$.

Let $x(\mathbf{c}, z, \theta)$ be the probability of receiving consumption $\mathbf{c}$ and being in price-island z, conditional upon the announcement of shock θ. Again, an agent can trade in the spot markets taking place in period $t = 1$ (trading c_1 today for c_2 tomorrow) to maximize her own utility, taking the contract and the spot price $p(z)$, or here the interest rate $r(z) = \frac{1}{p(z)}$, as given:

$$V(\mathbf{c}, z, \theta) = \max_{\tau_1, \tau_2} U(\mathbf{c}, \theta), \tag{49}$$

subject to the budget constraint

$$\tau_1 + p(z)\tau_2 = 0, \tag{50}$$

where again $p(z)$ is the market-clearing price of c_2 relative to c_1 when the market fundamental is z. Similar to the case of moral hazard, there is no loss of generality in focusing only on lotteries with no spot trade; that is, $x(\mathbf{c}, z, \theta) > 0$ only if $U(\mathbf{c}, \theta) = V(\mathbf{c}, z, \theta)$, and $x(\mathbf{c}, z, \theta) = 0$ otherwise. As a result, the market fundamental in price-island z is exactly z, and therefore the consistency constraints for the market fundamental will be omitted.

The probability constraint is now:

$$\sum_{\mathbf{c}, z} x(\mathbf{c}, z, \theta) = 1, \ \forall \theta. \tag{51}$$

Again, we ensure that the announcing shock θ is the true one by imposing the following IC constraint:

$$\sum_{\mathbf{c}, z} x(\mathbf{c}, z, \theta) U(\mathbf{c}, \theta) \geq \sum_{\mathbf{c}, z} x(\mathbf{c}, z, \theta') V(\mathbf{c}, z, \theta), \ \forall \theta, \theta'. \tag{52}$$

Note that with the possibility of retrading in spot markets at the specified price $p(z)$, the RHS of the IC constraint uses the indirect utility $V(\mathbf{c}, z, \theta)$, which depends on the market fundamental z. The resource constraint is given by:

$$\sum_{\mathbf{c}, \theta} \pi(\theta) x(\mathbf{c}, z, \theta) \left(c_1 + \frac{c_2}{R}\right) \leq \sum_{\mathbf{c}, \theta} \pi(\theta) x(\mathbf{c}, z, \theta) e, \ \forall z. \tag{53}$$

The *retrading-constrained optimal allocations with price-islands* are characterized by the following Pareto program.

Program 6.

$$\max_{x} \sum_{(\mathbf{c},z,\theta)} x\left(\mathbf{c},z,\theta\right) U(\mathbf{c},\theta) \tag{54}$$

subject to (51), (52), (53).

Again, this is a linear program, the solution of which exists and is the global maximum given the non-emptiness, compactness, and convexity of the constrained set, and the convexity and continuity of the objective function.

The competitive equilibrium with price-islands in this economy can be defined analogously to Definition 5 in Section 7. This is clearly a convex economy. Hence, the first and second welfare theorems and the existence theorem hold. The detail is omitted for brevity.

It is worth noting that the externality problem in this environment with unobserved states and retrading has been extensively studied in the literature [e.g., 16, 2, 11]. In particular, [11] show that a government intervention, for example a liquidity requirement, can solve the externality problem. However, the form of the intervention, either a liquidity floor or a liquidity cap, is sensitive to the preference structure. On the other hand, our competitive equilibrium with price-islands is retrading-constrained efficient under more general preferences, i.e., it is not sensitive to the preference structure.

10.2. Heterogeneity

The results in this paper also apply to an economy with ex ante heterogeneous agents, who may be endowed with different preferences or production technology. To be more precise, consider an economy with H types of agents, each of which has a continuum of agents with mass α^h such that $\sum_h \alpha^h = 1$. Each agent type h is endowed by utility function $U^h\left(\mathbf{c},a\right)$, and production technology $f^h\left(\mathbf{q}|a\right)$.

Let $x^h\left(a,\mathbf{c},\mathbf{q},z\right)$ be agent type h's probability of receiving recommended action a, receiving consumption $\mathbf{c}$, realizing output $\mathbf{q}$, and being in island z. Again, an agent can trade in the spot markets (of c_1 and c_2) to maximize her own utility, taking the contract and the spot price $p(z)$ as given. The *retrading-constrained optimal allocations with price-islands* are characterized by the following Pareto program.

Program 7.

$$\max_{\left(x^h\right)_h} \sum_{h} \lambda^h \alpha^h \sum_{(a,\mathbf{c},\mathbf{q},z)} x^h\left(a,\mathbf{c},\mathbf{q},z\right) U^h(\mathbf{c},a) \tag{55}$$

subject to

$$\sum_{(a,\mathbf{c},\mathbf{q},z)} x^h(a,\mathbf{c},\mathbf{q},z) = 1, \ \forall h \tag{56}$$

$$\sum_h \alpha^h \sum_{(a,\mathbf{c},\mathbf{q})} x^h(a,\mathbf{c},\mathbf{q},z)(\mathbf{q}-\mathbf{c}) \geq \mathbf{0}, \ \forall z \tag{57}$$

$$f^h(\mathbf{q}|a) \sum_{(\mathbf{c},\bar{\mathbf{q}},z)} x^h(a,\mathbf{c},\bar{\mathbf{q}},z) = \sum_{(\mathbf{c},z)} x^h(a,\mathbf{c},\mathbf{q},z), \ \forall a,\mathbf{q},h \tag{58}$$

$$\sum_{(\mathbf{c},\mathbf{q},z)} x^h(a,\mathbf{c},\mathbf{q},z) U^h(\mathbf{c},a) \geq \sum_{(\mathbf{c},\mathbf{q},z)} x^h(a,\mathbf{c},\mathbf{q},z) \frac{f^h(\mathbf{q}|\bar{a})}{f^h(\mathbf{q}|a)} V^h(\mathbf{c},\bar{a},z), \ \forall a,\bar{a},h \tag{59}$$

where $\left(\lambda^h\right)_h$ is the vector of Pareto weights such that $\sum_h \lambda^h = 1$. Note that the consumption possibility set of type h is now defined by constraints (56), (58), (59).

We now come back to the issue of allowing only one island when agents are ex ante heterogeneous. As discussed earlier in Section 8.1, [25] implies that having an infinite number of (heterogeneous) types can "aggregate out" the "jump" in the demand function, which causes a problem for the existence of a competitive equilibrium when we restrict attention to a unique, single price-island. The heterogeneity creates a new problem, however. In particular, each type may end up choosing a different price-island. That is, we may not be able to guarantee the uniqueness of the price-island using condition (46) for each type of agent. To be more precise, consider an economy with H types of agents, as discussed above. We now assume that each agent chooses island z before everything else; for each z and h,

$$\frac{x^h(a,\mathbf{c},\mathbf{q},z)}{\sum_z x^h(a,\mathbf{c},\mathbf{q},z)} = \frac{x^h(a',\mathbf{c}',\mathbf{q}',z)}{\sum_z x^h(a',\mathbf{c}',\mathbf{q}',z)}, \forall (a,\mathbf{c},\mathbf{q}), (a',\mathbf{c}',\mathbf{q}') \in A \times C \times Q. \tag{60}$$

Of course, Proposition 5 is still valid for each type h; that is, all agents of type h will end up in one island only. However, we can not guarantee that different types will choose the same islands. That is, condition (60) is not sufficient to ensure that there will be a single active island in an equilibrium.

11. Conclusion

This paper explicitly identifies the source of a market failure, an externality, when there is a private information problem and at the same time agents can retrade in ex post spot markets. The externality cannot be internalized under the contracting framework of [21, 22] unless preferences are partially separable. Nevertheless, allowing agents to contract ex ante on market fundamentals, coupling purchases and sales with the right to trade on endogenous exchanges, or price-islands, achieves a notion of constrained efficiency even with competitive ex post spot markets. One could view our results as normative, indicative of the need for more markets ex ante, not less, to solve externality problems that can cause problems in financial markets. Ironically though, the constrained efficient solution involves market segmentation ex post, i.e., winners and losers ex post, and those receiving different ex ante contracts, and not all mixed up with one another in one freely competitive market, but rather, are segregated into separate trading posts, each with its own price. Of course, implementing optimal contracts with segregated exchanges requires some enforcement, especially when there are multiple active exchanges in equilibrium (there will be no need for such commitment if there is only one active exchange).

Further, such externalities are likely to exist in dynamic environments with sequential private actions and we suspect there are similar ex ante market solutions. We believe that our market-based solution should be applicable to these issues as well. But this issue is beyond the scope of the current paper and we leave it as a subject for future research.

A. More Proofs

Lemma 2. *A constrained optimal allocation without retrading will equate marginal rates of substitution across agents if (17) holds for any $(a, \mathbf{c}, \mathbf{q})$ with $x(a, \mathbf{c}, \mathbf{q}) > 0$.*

Proof. To prove this result, consider the first-order condition of Program 2 with respect to $x(a, \mathbf{c}, \mathbf{q})$:

$$U(\mathbf{c}, a) + \mu_l + \sum_i \mu_{rc}(i)\,(q_i - c_i) + \sum_{\bar{\mathbf{q}}} \left[\mu_{mn}(a, \bar{\mathbf{q}}) - \mu_{mn}(a, \mathbf{q})\right] f(\bar{\mathbf{q}}|a) + \sum_{a'} \mu_{ic}(a, a') \left[U(\mathbf{c}, a) - U(\mathbf{c}, a')\frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)}\right] \leq 0, \tag{61}$$

where it holds with equality if $x(a, \mathbf{c}, \mathbf{q}) > 0$, and $\mu_l, \mu_{rc}(i), \mu_{mn}(a, \mathbf{q}), \mu_{ic}(a, a')$ are the Lagrange multipliers for the probability constraint (2), the resource constraint of good i (5), the mother-nature constraint for $(a, \mathbf{q})$ (3), and the incentive compatibility constraint for (a, a') (14), respectively.

We now focus on the first-order condition (61) that holds with equality (i.e., $x(a, \mathbf{c}, \mathbf{q}) > 0$). For simplicity, we can imagine that the grids for consumption allocations are so fine that we can take derivative with respect to each of them. Differentiating (61) with respect to c_1 and c_2, respectively, gives

$$U_1(\mathbf{c}, a) + \sum_{a'} \mu_{ic}(a, a') \left[U_1(\mathbf{c}, a) - U_1(\mathbf{c}, a')\frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)}\right] = \mu_{rc}(1), \tag{62}$$

$$U_2(\mathbf{c}, a) + \sum_{a'} \mu_{ic}(a, a') \left[U_2(\mathbf{c}, a) - U_2(\mathbf{c}, a')\frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)}\right] = \mu_{rc}(2), \tag{63}$$

where $U_i(\mathbf{c}, a) \equiv \frac{\partial U(\mathbf{c}, e)}{\partial c_i}$ is the marginal utility with respect to good i. These conditions can be rewritten as

$$U_1(\mathbf{c}, a) \left[1 + \sum_{a'} \mu_{ic}(a, a') \left[1 - \frac{U_1(\mathbf{c}, a')}{U_1(\mathbf{c}, a)}\frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)}\right]\right] = \mu_{rc}(1), \tag{64}$$

$$U_2(\mathbf{c}, a) \left[1 + \sum_{a'} \mu_{ic}(a, a') \left[1 - \frac{U_2(\mathbf{c}, a')}{U_2(\mathbf{c}, a)}\frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)}\right]\right] = \mu_{rc}(2). \tag{65}$$

Dividing (65) by (64) gives

$$\left(\frac{U_2(\mathbf{c},a)}{U_1(\mathbf{c},a)}\right)\left(\frac{1+\sum_{a'}\mu_{ic}(a,a')\left[1-\frac{U_2(\mathbf{c},a')}{U_2(\mathbf{c},a)}\frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)}\right]}{1+\sum_{a'}\mu_{ic}(a,a')\left[1-\frac{U_1(\mathbf{c},a')}{U_1(\mathbf{c},a)}\frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)}\right]}\right) = \frac{\mu_{rc}(2)}{\mu_{rc}(1)}. \tag{66}$$

This equation implies that $\frac{U_2(\mathbf{c},a)}{U_1(\mathbf{c},a)}$ will be the same as $\frac{\mu_{rc}(2)}{\mu_{rc}(1)}$ for any $(a,\mathbf{c},\mathbf{q})$ with $x(a,\mathbf{c},\mathbf{q})>0$ if the second fraction on the LHS is equal to 1:

$$\sum_{a'}\mu_{ic}(a,a')\left[1-\frac{U_2(\mathbf{c},a')}{U_2(\mathbf{c},a)}\frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)}\right]=\sum_{a'}\mu_{ic}(a,a')\left[1-\frac{U_1(\mathbf{c},a')}{U_1(\mathbf{c},a)}\frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)}\right], \tag{67}$$

which in turn implies that

$$\sum_{a'}\mu_{ic}(a,a')\left[\frac{U_2(\mathbf{c},a')}{U_2(\mathbf{c},a)}-\frac{U_1(\mathbf{c},a')}{U_1(\mathbf{c},a)}\right]\frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)}=0. \tag{68}$$

This condition can be further rearranged as

$$\sum_{a'}\mu_{ic}(a,a')\frac{U_1(\mathbf{c},a')}{U_2(\mathbf{c},a)}\frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)}\left[\frac{U_2(\mathbf{c},a')}{U_1(\mathbf{c},a')}-\frac{U_2(\mathbf{c},a)}{U_1(\mathbf{c},a)}\right]=0. \tag{69}$$

Q.E.D.

Proof of Theorem 8.1. Let (x,y), and $P(a,\mathbf{c},\mathbf{q},z)$ be a competitive equilibrium with price-islands. Suppose the competitive equilibrium allocation is not retrading-constrained optimal, i.e., there is a feasible allocation $\tilde{x}$ such that $\sum_{a,\mathbf{c},\mathbf{q},z}\tilde{x}(a,\mathbf{c},\mathbf{q},z)U(\mathbf{c},a)>\sum_{a,\mathbf{c},\mathbf{q},z}x(a,\mathbf{c},\mathbf{q},z)U(\mathbf{c},a)$. With local nonsatiation of preferences, we have

$$\sum_{a,\mathbf{c},\mathbf{q},z}P(a,\mathbf{c},\mathbf{q},z)\,x(a,\mathbf{c},\mathbf{q},z)<\sum_{a,\mathbf{c},\mathbf{q},z}P(a,\mathbf{c},\mathbf{q},z)\,\tilde{x}(a,\mathbf{c},\mathbf{q},z). \tag{70}$$

Using the market-clearing condition (42), we can replace $x(a,\mathbf{c},\mathbf{q},z)$ on the LHS of (70) with $y(a,\mathbf{c},\mathbf{q},z)$. As a result, the LHS of (70) becomes the profit of the broker-dealer, which is zero.

On the other hand, the RHS of (70) can be rewritten as

$$\begin{aligned}\sum_{a,\mathbf{c},\mathbf{q},z}P(a,\mathbf{c},\mathbf{q},z)\,\tilde{x}(a,\mathbf{c},\mathbf{q},z) &\le \sum_{a,\mathbf{c},\mathbf{q},z}\sum_i\widetilde{P}_i(z)\,(c_i-q_i)\,\tilde{x}(a,\mathbf{c},\mathbf{q},z)\\ &= \sum_z\sum_i\widetilde{P}_i(z)\sum_{a,\mathbf{c},\mathbf{q}}\tilde{x}(a,\mathbf{c},\mathbf{q},z)\,(c_i-q_i)=0,\end{aligned} \tag{71}$$

where the first inequality follows from the optimum condition for profit maximization (41), and the last equality follows from the resource constraint for each z (34). Therefore, (70) now becomes $0<0$. This is a contradiction! *Q.E.D.*

Proof of Theorem 8.2. Given that the optimization problems are well-defined concave problems, Kuhn-Tucker conditions are necessary and sufficient. The proof is divided into three steps:

(i) Kuhn-Tucker conditions for Pareto Optimal allocations: We will first characterize a solution to the Pareto program using Kuhn-Tucker conditions. Let μ_l, $\mu_{rc}(i,z)$, $\mu_{mn}(a,\mathbf{q})$, $\mu_{ic}(a,a')$ be the Lagrange multipliers for the probability constraint (30), the resource constraint of good i in price-island z (34), the mother-nature constraint for $(a,\mathbf{q})$ (31), and the incentive compatibility constraint for (a,a') (32), respectively. All non-negativity constraints are kept implicit for brevity. A solution to the Pareto program satisfies the following condition, for each $x(a,\mathbf{c},\mathbf{q},z)$,

$$U(\mathbf{c},a)+\mu_l+\sum_i \mu_{rc}(i,z)(q_i-c_i)+\sum_{\mathbf{q}'}\left[\mu_{mn}(a,\mathbf{q}')-\mu_{mn}(a,\mathbf{q})\right]f(\mathbf{q}'|a)$$
$$+\sum_{a'}\mu_{ic}(a,a')\left[U(\mathbf{c},a)-V(\mathbf{c},a',z)\frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)}\right]\le 0, \tag{72}$$

where the inequality holds with equality if $x(a,\mathbf{c},\mathbf{q},z)>0$.

(ii) Kuhn-Tucker conditions for equilibrium allocations: We will characterize solutions to the consumers' and broker-dealers' problems in equilibrium using Kuhn-Tucker conditions. Let $\nu_l,\nu_{mn}(a,\mathbf{q}),\nu_{ic}(a,a')$, and ν_u be the Lagrange multipliers for the probability constraint (30), the mother-nature constraint for $(a,\mathbf{q})$ (31), the incentive compatibility constraint for (a,a') (32), and the participation constraint (45) respectively. All non-negativity constraints are kept implicit for brevity. The optimal condition for $x(a,\mathbf{c},\mathbf{q},z)$ is given by

$$\nu_u U(\mathbf{c},a)+\nu_l-P(a,\mathbf{c},\mathbf{q},z)+\sum_{\mathbf{q}'}\left[\nu_{mn}(a,\mathbf{q}')-\nu_{mn}(a,\mathbf{q})\right]f(\mathbf{q}'|a) \tag{73}$$
$$+\sum_{a'}\nu_{ic}(a,a')\left[U(\mathbf{c},a)-V(\mathbf{c},a',z)\frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)}\right]\le 0, \tag{74}$$

where the inequality holds with equality if $x(a,\mathbf{c},\mathbf{q},z)>0$. Recall that the optimal condition of the broker-dealer's profit maximization problem (40), for each contract $(a,\mathbf{c},\mathbf{q},z)$, is

$$P(a,\mathbf{c},\mathbf{q},z)\le\sum_i \widetilde{P}_i(z)(c_i-q_i), \tag{75}$$

where the condition holds with equality if $y(a,\mathbf{c},\mathbf{q},z)>0$.

(iii) Matching dual variables and prices: We will show that the optimal conditions of the Pareto program are equivalent to the optimal conditions of consumers' and broker-dealer's problems. Recall that good-1 is the numeraire. Let $\mu_l=\frac{\nu_l}{\nu_u},\mu_{rc}(i,z)=\frac{\widetilde{P}_i(z)}{\nu_u},\mu_{mn}(a,\mathbf{q})=\frac{\nu_{mn}(a,\mathbf{q})}{\nu_u},\mu_{ic}(a,a')=\frac{\nu_{ic}(a,a')}{\nu_u}$. Using the matching conditions specified above, the optimal condition for the constrained optimality (72) becomes

$$\nu_u U(\mathbf{c},a)+\nu_l-\sum_i \widetilde{P}_i(z)(c_i-q_i)+\sum_{\mathbf{q}'}\left[\nu_{mn}(a,\mathbf{q}')-\nu_{mn}(a,\mathbf{q})\right]f(\mathbf{q}'|a) \tag{76}$$
$$+\sum_{a'}\nu_{ic}(a,a')\left[U(\mathbf{c},a)-V(\mathbf{c},a',z)\frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)}\right]\le 0,$$

where the inequality holds with equality if $x(a, \mathbf{c}, \mathbf{q}, z) > 0$.

On the other hand, using (75), the optimal condition for the equilibrium (73) becomes

$$\nu_u U(\mathbf{c}, a) + \nu_l - \sum_i \widetilde{P}_i(z)(c_i - q_i) + \sum_{\mathbf{q}'} [\nu_{mn}(a, \mathbf{q}') - \nu_{mn}(a, \mathbf{q})] f(\mathbf{q}'|a) + \sum_{a'} \nu_{ic}(a, a') \left[U(\mathbf{c}, a) - V(\mathbf{c}, a', z) \frac{f(\mathbf{q}|a')}{f(\mathbf{q}|a)} \right] \leq 0, \tag{77}$$

where the inequality holds with equality if $x(a, \mathbf{c}, \mathbf{q}, z) > 0$. This condition is exactly the same as (76). This shows that a solution to the Pareto program also solves the consumers' and broker-dealers' problems.

Recall that the resource constraints in the Pareto program are identical to the market-clearing conditions in equilibrium. Hence, we have shown that any retrading-constrained optimal allocation is also a compensated equilibrium allocation.

We now show that the equilibrium expenditure of x is zero, i.e., $\sum_{a,\mathbf{c},\mathbf{q},z} P(a, \mathbf{c}, \mathbf{q}, z) x(a, \mathbf{c}, \mathbf{q}, z) = 0$. To prove this, consider a complementarity slackness of the resource constraints (34):

$$\mu_{rc}(i, z) \left[\sum_{a,\mathbf{c},\mathbf{q}} x(a, \mathbf{c}, \mathbf{q}, z)(q_i - c_i) \right] = 0.$$

Summing this equation over i and z gives:

$$\begin{aligned} 0 &= \sum_i \mu_{rc}(i, z) \left[\sum_{a,\mathbf{c},\mathbf{q}} x(a, \mathbf{c}, \mathbf{q}, z)(q_i - c_i) \right] = \sum_{a,\mathbf{c},\mathbf{q},z} x(a, \mathbf{c}, \mathbf{q}, z) \sum_i \mu_{rc}(i, z)(q_i - c_i) \\ &= \sum_{a,\mathbf{c},\mathbf{q},z} x(a, \mathbf{c}, \mathbf{q}, z) \sum_i \frac{\widetilde{P}_i(z)}{\nu_u}(q_i - c_i) = \frac{1}{\nu_u} \sum_{a,\mathbf{c},\mathbf{q},z} P(a, \mathbf{c}, \mathbf{q}, z) x(a, \mathbf{c}, \mathbf{q}, z), \end{aligned}$$

where the third equality uses the matching condition above, and the last equality follows from (41). This clearly implies that the equilibrium expenditure $\sum_{a,\mathbf{c},\mathbf{q},z} P(a, \mathbf{c}, \mathbf{q}, z) x(a, \mathbf{c}, \mathbf{q}, z) = 0$. *Q.E.D.*

Proof of Theorem 8.3. Let x be a retrading-constrained optimal allocation. According to Theorem 8.2, any retrading-constrained optimal allocation can be supported as a compensated equilibrium, and the equilibrium expenditure of x is zero. Hence, we only need to show that any compensated equilibrium is a competitive equilibrium with price-islands. In particular, we will use a cheaper-point argument to show that the expenditure minimization (44) is equivalent to the utility maximization (36).

In order to do so, we shall show that there exists an allocation $\widehat{x} \in X$ that costs less than x. An Inada condition ($\lim_{c \to 0} U_i(\mathbf{c}, a) = \infty$ for $i = 1, 2$) guarantees that a solution to the Pareto program 4, which is a compensated equilibrium allocation, will not have a strictly positive mass on $c = 0$. Let $0 \in C$; that is, the zero consumption allocation is on the grid.

Consider an alternative allocation, $\hat{x}$, that puts all mass on contracts with zero consumption and the least costly action $\underline{a}$:

$$\hat{x}\left(\underline{a},0,\mathbf{q},z\right) = \sum_{a,\mathbf{c}} x\left(a,\mathbf{c},\mathbf{q},z\right),\ \forall \mathbf{q},z, \tag{78}$$

$$\hat{x}\left(a,\mathbf{c},\mathbf{q},z\right) = 0,\ \text{if } a \neq \underline{a} \text{ or } \mathbf{c} \neq 0. \tag{79}$$

It is not difficult to show that $\hat{x}$ satisfies the probability and the mature-nature constraints (30)-(31) using the fact that x also satisfies both constraints. The IC constraint holds because the alternative lottery requires the lowest action, and the compensation is zero regardless. Therefore, there is no incentive to deviate toward a higher action. In summary, $\hat{x}$ is feasible. We only now need to show that it costs less than x.

The optimal condition of the broker-dealer (41) implies that, for a given bundle $(\mathbf{q},z)$,

$$P\left(\underline{a},0,\mathbf{q},z\right) < P\left(a,\mathbf{c},\mathbf{q},z\right),\ \forall a,\ \text{and } \mathbf{c} \neq 0. \tag{80}$$

Hence, the cost of $\hat{x}$ is:

$$\begin{aligned}
\sum_{a,\mathbf{c},\mathbf{q},z} P\left(a,\mathbf{c},\mathbf{q},z\right)\hat{x}\left(a,\mathbf{c},\mathbf{q},z\right) &= \sum_{\mathbf{q},z} P\left(\underline{a},0,\mathbf{q},z\right)\hat{x}\left(\underline{a},0,\mathbf{q},z\right) \\
&= \sum_{\mathbf{q},z} P\left(\underline{a},0,\mathbf{q},z\right)\sum_{a,\mathbf{c}} x\left(a,\mathbf{c},\mathbf{q},z\right) \\
&< \sum_{\mathbf{q},z} P\left(a,\mathbf{c},\mathbf{q},z\right)\sum_{a,\mathbf{c}} x\left(a,\mathbf{c},\mathbf{q},z\right) \\
&= \sum_{a,\mathbf{c},\mathbf{q},z} P\left(a,\mathbf{c},\mathbf{q},z\right)x\left(a,\mathbf{c},\mathbf{q},z\right),
\end{aligned}$$

where the first equality follows from (79), the second equality uses (78), and the inequality follows from (80). This shows that there exists a feasible allocation $\hat{x}$ that is cheaper than the compensated equilibrium allocation, x. As a result, using the cheaper-point argument, a compensated equilibrium is a competitive equilibrium with price-islands. *Q.E.D.*

Proof of Proposition 5. The proof is a contradiction argument. Suppose there are at least two active islands, $z \neq z'$, i.e., $\sum_{a,\mathbf{c},\mathbf{q}} x\left(a,\mathbf{c},\mathbf{q},z\right) > 0$ and $\sum_{a,\mathbf{c},\mathbf{q}} x\left(a,\mathbf{c},\mathbf{q},z'\right) > 0$. For brevity, the proof will be written in terms of conditional probability, $Pr(\cdot|\cdot)$, which can be written in terms of x if needed.

Condition (46) can be rewritten in terms of $Pr\left(z|a,\mathbf{c},\mathbf{q}\right)$ as, for any z,

$$Pr\left(z|a,\mathbf{c},\mathbf{q}\right) = Pr\left(z|a',\mathbf{c}',\mathbf{q}'\right) \Longrightarrow \frac{Pr\left(z|a,\mathbf{c},\mathbf{q}\right)}{Pr\left(z|a',\mathbf{c}',\mathbf{q}'\right)} = 1,\ \forall\left(a,\mathbf{c},\mathbf{q}\right),\left(a',\mathbf{c}',\mathbf{q}'\right) \in A \times C \times Q.$$

This implies that, for any z and z',

$$\frac{Pr\left(z|a,\mathbf{c},\mathbf{q}\right)}{Pr\left(z|a',\mathbf{c}',\mathbf{q}'\right)} = \frac{Pr\left(z'|a,\mathbf{c},\mathbf{q}\right)}{Pr\left(z'|a',\mathbf{c}',\mathbf{q}'\right)}. \tag{81}$$

Multiplying both sides by a common term, $\frac{Pr(a,\mathbf{c},\mathbf{q})}{Pr(a',\mathbf{c}',\mathbf{q}')}$, gives

$$\begin{aligned}\frac{Pr\left(a,\mathbf{c},\mathbf{q}\right)Pr\left(z|a,\mathbf{c},\mathbf{q}\right)}{Pr\left(a',\mathbf{c}',\mathbf{q}'\right)Pr\left(z|a',\mathbf{c}',\mathbf{q}'\right)} &= \frac{Pr\left(a,\mathbf{c},\mathbf{q}\right)Pr\left(z'|a,\mathbf{c},\mathbf{q}\right)}{Pr\left(a',\mathbf{c}',\mathbf{q}'\right)Pr\left(z'|a',\mathbf{c}',\mathbf{q}'\right)} \\ &\Longrightarrow \frac{Pr\left(a,\mathbf{c},\mathbf{q},z\right)}{Pr\left(a',\mathbf{c}',\mathbf{q}',z\right)} = \frac{Pr\left(a,\mathbf{c},\mathbf{q},z'\right)}{Pr\left(a',\mathbf{c}',\mathbf{q}',z'\right)},\end{aligned}$$

for all $(a,\mathbf{c},\mathbf{q}),(a',\mathbf{c}',\mathbf{q}') \in A \times C \times Q$. Using Bayes' rule, this can be written as

$$\frac{Pr\left(a,\mathbf{c},\mathbf{q}|z\right)}{Pr\left(a',\mathbf{c}',\mathbf{q}'|z\right)} = \frac{Pr\left(a,\mathbf{c},\mathbf{q}|z'\right)}{Pr\left(a',\mathbf{c}',\mathbf{q}'|z'\right)},\ \forall\left(a,\mathbf{c},\mathbf{q}\right),\left(a',\mathbf{c}',\mathbf{q}'\right) \in A \times C \times Q. \tag{82}$$

As probability measures, $\sum_{a,\mathbf{c},\mathbf{q}} Pr\left(a,\mathbf{c},\mathbf{q}|z\right) = \sum_{a,\mathbf{c},\mathbf{q}} Pr\left(a,\mathbf{c},\mathbf{q}|z'\right) = 1$. Using this fact, condition (82) now becomes

$$Pr\left(a,\mathbf{c},\mathbf{q}|z\right) = Pr\left(a,\mathbf{c},\mathbf{q}|z'\right),\forall\left(a,\mathbf{c},\mathbf{q}\right) \in A \times C \times Q. \tag{83}$$

We now argue that the last equality above implies that the market fundamentals in two islands, z and z', are identical. Note that $Pr\left(a,\mathbf{c},\mathbf{q}|z\right)$ can be interpreted as a distribution of ex-post (but before retrading) endowment in island z, which in turns determines the market fundamental. Accordingly, (83) implies that the distribution of ex-post allocation in both islands are identical. Therefore, by definition, the market fundamentals in two islands, z and $\tilde{z}$, must be the same. This is a contradiction.

Since any feasible lottery puts positive mass on one island only, we can now conclude that a solution to Program 5 must have only one active island. *Q.E.D.*

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References

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